Information flows as a path least resistance: A category theory approach using preradicals

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Abstract. Category theory has been recently used as a tool for constructing and modeling an information flow framework. Here, we use the theory of preradicals to show that the flow of information can be described using preradicals and its properties. We show that preradicals generalize the notion of persistence to spaces where the underlying structure forms a directed acyclic graph. We prove that the persistence module associated with a directed acyclic graph can be obtained by a particular α preradical. Given how preradicals are defined, they can be considered as compatible choice assignments that preserve the underlying structure of the modeled system. This drives us to generalize the notions of standard persistence, zigzag persistence, and multidirectional persistence.

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1 Introduction

This is the first of two related papers that aim to build a formal framework for the conceptual axiom that in a general sense the flow of information through a network, independent of the physical processes that carry that information in any specific network, will always follow a path of 'least resistance' that is ultimately responsible for the consequential dynamics on the network. While distinct physical processes are responsible for driving this process, we propose that a least resistive information flow path is a natural consequence and universal phenomenon. The critical intuition here is that while *a priori* descriptive models or algorithmic rules may set up the conditions for the dynamics of a network, it is the flow of signals and information through the network bounded by physical constraints that dictate how computational events are prioritized and executed, by following a path of least resistance that are dependent on such constraints. Yet, to the best of our knowledge, a theoretical framework that captures the universality of this effect, independent of any physical details, but which also allows computing and predicting what those information flow least resistive pathways are in real networks, does not exist.

For example, in a literal way, water flowing through a set of interconnected pipes will flow down a path of least resistance as dictated by the pressure fronts associated with the flow. The physics responsible for this include the incompressibility of water and the compliance and diameter of the pipes in the network. The resultant effect is a 'natural' flow dictated and constrained by the physics that takes the route of least effort. Weight changes in artificial neural networks are shaped by constraints, i.e. their own version of the physics that makes them up, imposed by the connectivity structure of the network, error or loss functions, and the algorithms (rules) that affect weight changes, e.g. backpropagation. The consequence of these constraints and rules are decentralized but integrated weight changes that allow the network to properly classify new inputs. The weight changes themselves however, follow a flow of least effort or resistance that is a natural consequence of the physical make up and rules of the network. Another example are real biological neural networks composed of interconnected neurons. In fact, this is our main practical motivation for this work. There are myriad of quantiative models that describe different aspects of neural dynamics and the structural rules that specify how brain networks are connected. Yet, describing how internal computations and information flows are prioritized is not possible, regardless of the model. We do not fully understand the foundational principles that govern how networks of neurons successfully operate autonomously. What the drivers are that shape the prioritization, order, and decisions of the internal computations and operations after a network receives an input. And how those computations produce actionable information that they pass on to other neurons. This is distinct from specifying an algorithm a network needs to follow or a model of the network. Mechanistically in the case of biological neural networks, an example of a fundamental physical process that impose such constraints are structure (connectivity and geometry)-function constraints and energy considerations [17]. In human social networks, we tend to make friends with individuals we relate to and like after introductions are made. In other words, again, network connectivity is one physical constraint that sets the condition for the creation of pathways that reflect least effort or resistance, i.e. individuals we like. In many systems, including artificial neural networks and real biological brain networks, these processes are responsible for the passing of information from one scale of organization or hierarchy to another.

In this work, we rely on category theory to describe how information flows through a system that can be modeled using objects and morphisms in the category K-Mod, where K is a field. Our approach allows us to show that the flow of information is preserved within the underlying structure of the modeled system. We address this by using a theory that involves the use of persistent modules, which derive from persistence homology. Persistent homology considers a family of topological spaces and inclusions from one space into another to find topological features common to subsets of these spaces. The persistent homology is defined by homology groups that allow classifying invariant topological features. We can think, for example, of the standard persistent homology where spaces are linearly ordered with one space included in the next. Similarly, we can think of zigzag persistence [5], where the spaces are linearly ordered but the inclusions can occur in either direction. Another example is multidimensional persistence [7] which operates in multiple dimensions on a grid with inclusion maps parallel to the coordinate axis.

In [8] Chambers and Letsher extended these notions of persistence by considering that the underlying structure form a directed acyclic graph (DAG), naming it DAG persistence. These authors used an algebraic representation called *G-module* which have a commutative condition that appear from following distinct paths in a DAG between a same pair of vertices. It is using this

G-module structure where DAG persistence gets common subgroups to all of the homology groups. This representation construction is based on algebraic structures (i.e. vector spaces) that can be considered as objects in the category of K-modules for K a field. In this paper, we will describe persistence from a category theory perspective, using a tool known as preradicals. Preradicals produce compatible substructures within each object that are preserved by morphisms, allowing for an efficient determination of algebraic invariants. Using preradicals that we show how information flows through the G-module representation.

In this paper, we generalize the notion of persistence by showing that the persistence of a G-module, as defined in [8], can be obtained using a particular type of preradical in the category K-Mod. This generalization allows us to use preradicals to examine how information is transmitted between the objects that comprise the G-module representation in such a way that the underlying structure is preserved.

The paper is organized as follows: in Section 2 we present preliminary definitions for directed acyclic graphs and quivers. We then give the basic construction of a quiver representation and the definition of persistence in a G-module. In Section 3 we introduce the definition of preradicals with the four principal operations between them. We discuss the role of preradicals as a tool to describe the way information is transmitted within the category, such that the structure is preserved. We conclude this section with the definition of the α and ω preradicals. Our main contributions are introduced in Section 4, where we show that the persistence of a G-module whose underlying structure is a single-source single-sink graph, is obtained by the preradical $\alpha^{M_s}_{M_s}$. We then generalize this proposition to a G-module whose underlying structure is a graph with n sources and m sinks. Section 5 discuss the role of preradicals as a way for describing the flow of information. Section 6 provides some concluding arguments. Finally, in Appendix A we provide a brief description of the construction of homology groups as well as the persistence homology group; whereas in Appendix B we mention different categories where the G-module representation of a DAG can be consider, and whose decomposition theorems are still valid.

2 Preliminaries

We begin by giving preliminary definitions of relevance to the rest of our paper. For a complete introduction to homology groups we recommend reading some of standard references [3], [6], [11] and [16]. Most models of persistence use a collection of spaces and inclusions of one space into another to find topological features common to subsets among these spaces. In [8] they show a generalization that considers inclusions over a set of spaces that form a directed graph, with the constrains that the graph must be a directed acyclic graph (DAG): acyclic and not contain repeated edges.

Definition 2.1. *For a simple directed acyclic graph $G = (V, E)$, a graph filtration χ_G of a topological space X is a pair $(\{X_v\}_{v\in V}, \{f_e\}_{e\in E})$ such that*

(1) $X_v \subset X$ for all $v \in V$;

(2) If $e = (v, u) \in E$ then $f_e : X_v \longrightarrow X_u$ is a continuous embedding (or inclusion) of X_v into X_u.

Thus, the persistence group of a filtration in a topological space can be considered as the subgroups common to all of the homology groups.

A *quiver* is a directed graph where loops and multiple directed edges between the same vertices are allowed. Formally, a quiver is quadruple $Q = (V, E, s, t)$ where V is the set of *vertices*, E the set of *edges* and $s, t : E \longrightarrow V$ are two maps, assigning the *starting vertex* and the *ending vertex* for each edge. In this case, for an edge e with $s(e) = u$ and $t(e) = v$ we write $e : u \longrightarrow v$. A quiver Q is finite if both sets V and E are finite. In particular, every DAG is a quiver.

Typically quivers admit a *representation* over a field K which assigns to each vertex v a vector space W_v and to each arrow $e : u \longrightarrow v$ a K-linear morphism map $f_e : W_u \longrightarrow W_v$. More formally,

Definition 2.2. *Given a quiver $Q = (V, E, s, t)$, a representation of Q over a field K is a pair of families*

$$\Big(\{W_v\}_{v\in V}, \{f_e\}_{e\in E}\Big)$$

where for each arrow $e : u :\longrightarrow v$, $f_e : W_u \longrightarrow W_v$ is a linear morphism.

If $M = \Big(\{W_v\}_{v\in V}, \{f_e\}_{e\in E}\Big)$ and $M' = \Big(\{W'_v\}_{v\in V}, \{f'_e\}_{e\in E}\Big)$ are two representations of a quiver Q, a *morphism* $\gamma : M \longrightarrow M'$ is a family of linear morphisms $\{\gamma_v : W_v \longrightarrow W'_v\}_{v\in V}$ such that the diagram

$$\begin{array}{ccc} W_u & \xrightarrow{f_e} & W_v \\ {\scriptstyle\gamma_u}\downarrow & & \downarrow{\scriptstyle\gamma_v} \\ W'_u & \xrightarrow{f'_e} & W'_v. \end{array}$$

commutes for any $e \in E$. The composition of morphisms $\gamma : M \longrightarrow M'$ and $\beta : M' \longrightarrow M''$ is the morphisms $(\beta \circ \gamma) : M \longrightarrow M''$ defined by the family of compositions $\{\beta_v \circ \gamma_v\}_{v\in V}$, which is clearly associative and has as identity element $Id_M := \{Id_{V_i}\}_{i\in V}$. This shows that the collection of all representations for a quiver Q along with the composition operation forms a category, which we denote by *Rep*(Q).

A representation of a quiver Q is referred to as a G-module (see [1]). Since we are interested in quivers with relations, specifically commutative conditions, we add the following condition:

Definition 2.3. *The diagrams commutes: for any path $\gamma = e_1, \cdots, e_n$ in a quiver Q, one can extend this to a function $f_\gamma = f_{e_n} \circ \cdots \circ f_{e_1}$ in the G-module. Then, given γ and γ' two different directed paths in Q connecting vertices u and v, commutativity means $f_\gamma = f_{\gamma'}$ in the G-module.*

The next definition provides the framework to generalize the notion of zigzag persistence, and the multidirectional persistence, which will be shown in Section 4.

Definition 2.4. *[8, Definition 2.3] For a directed acyclic graph $G = (V, E)$ a k-dimensional persistence module for a graph filtration χ_G, is the commutative G-module $(\{W_v\}_{v\in V}, \{f_e\}_{e\in E})$ where*

(i) $W_v = H_k(X_v)$ for all $v \in V$;

(ii) For every edge $e = (u, v) \in E$, $f_e : H_k(X_u) \longrightarrow H_k(X_v)$ is the map induced by the the inclusion $X_u \hookrightarrow X_v$.

Suppose now we are given a directed acyclic graph $G = (V, E)$ and its commutative G-module representation, denoted as $M = (\{W_v\}_{v\in V}, \{f_e\}_{e\in E})$. A cone in the category K-Mod for the G-module M is a pair $(L, \{\eta_u\}_{u\in V})$ where L is a vector space and η_u is a linear morphism for each $u \in V$, such that for any edge $e = (u, v)$, we have the following commutative diagram

$$\begin{array}{ccccc} & & L & & \\ & {\scriptstyle\eta_u}\swarrow & & \searrow{\scriptstyle\eta_v} & \\ W_u & & \xrightarrow{\quad f_e \quad} & & W_v \end{array} ,$$

that is, $f_e \circ \eta_u = \eta_v$. Thus, a limit for the G-module is a cone with the universal property: for any other cone $(L', \{\eta'_u\}_{u \in V})$ there exist a unique morphism $\eta : L' \longrightarrow L$ such that $\eta_u \circ \eta = \eta'_u$ for every $u \in V$:

$$\begin{array}{ccc} & L' & \\ {}^{\eta}\swarrow & & \searrow^{\eta'_u} \\ L & \xrightarrow{\eta_u} & W_u \end{array} .$$

A cocone for the G-module M is defined in a similar way, as the pair $(L, \{\eta_u\}_{u \in V})$ where L is a vector space and η_u is a linear morphisms for each $u \in V$, such that for any edge $e = (u, v)$, the following diagram commutes

$$\begin{array}{ccc} W_u & \xrightarrow{f_e} & W_v \\ {}_{\eta_u}\searrow & & \swarrow_{\eta_v} \\ & L & \end{array} ,$$

A colimit for the G-module M is a cocone with the universal property: for any other cocone $(L', \{\eta'_u\}_{u \in V})$ there exist a unique morphism $\eta : L \longrightarrow L'$ such that $\forall u \in V$, the diagram

$$\begin{array}{ccc} & W_u & \\ {}^{\eta_u}\swarrow & & \searrow^{\eta'_u} \\ L & \xrightarrow{\eta} & L' \end{array} .$$

is commutative.

The category K-Mod is known to be complete and cocomplete, that is, where all small limits and colimits exists. Since commutative diagrams are small categories, the limit and colimit always exist for a commutative G-module. Therefore, by the commutativity of the G-module M and the categorical properties of the limit and colimit, we have an induced map

$$\eta_M : \mathit{lim}(M) \longrightarrow \mathit{colim}(M).$$

This is the precursor to defining the persistence of a G-module.

Definition 2.5. *(Definition 2.5 in [8].) If M is a commutative G-module then the persistence of M, denoted by $P(M)$, is the image $\eta_M(\mathit{lim}(M))$.*

The persistence $P(M)$ of a G-module M tells us how much information of $\mathit{lim}(M)$ persists through the G-module structure but not about the information that persists from each component of the G-module. In this context, preradicals can provide information about the evolution of the substructures of $\eta_M(\mathit{lim}(M))$ as well as the substructures from each component, along the G-module representation. On the one hand, by applying a preradical σ to $\mathit{lim}(M) \xrightarrow{\eta_M} \mathit{colim}(M)$ we can get the substructure defined by σ that is preserved in the G-module. On the other hand, by applying a preradical σ to each component of the G-module representation, we obtain a blue print of the information defined by σ that is present in the entire G-module structure.

3 Preradicals

In this section we consider the role of preradicals as a way to describe how information is transmitted within the objects and arrows that comprise the category of R-Mod. For a complete introduction to preradicals and its properties in R-Mod see [2], [14] and [15]. Preradicals are functors that assign to each object a subobject in such a way that for any morphism between two objects, the restriction and corestriction of the morphisms to the respective subojects are preserved. In other words, preradicals behave as subfunctors of the identity functor.

Example. *Consider the category $\mathbb{Z}$-Mod of all $\mathbb{Z}$-modules. This category is isomorphic to the category Ab of all abelian groups. Now, given $M \in \mathbb{Z}$-Mod, one can define*

$$\sigma(M) = \{x \in M \mid 2x = 0\}.$$

Notice first that $\sigma(M)$ is a submodule of M. Also, for any morphism $f : M \longrightarrow M'$ in $\mathbb{Z}$-Mod and any $y \in M$, we have that $f(2y) = 2f(y)$. Thus, if $x \in \sigma(M)$ we have that $2f(x) = f(2x) = 0$, which implies that $f(x) \in \sigma(M')$. Hence, $f(\sigma(M)) \subseteq \sigma(M')$ which in turns implies that

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & M' \\ {\scriptstyle \iota}\uparrow & & \uparrow{\scriptstyle \iota} \\ \sigma(M) & \xrightarrow{\ f_{|}\ } & \sigma(M'). \end{array}$$

is a commutative diagram.

The above example shows an endofunctor [1] on $\mathbb{Z}$-Mod which acts as a subfunctor of the identity functor on $\mathbb{Z}$-Mod. This is precisely the definition of a preradical:

Definition 3.1. *Let $\mathcal{C}$ be a category. A preradical σ on the category $\mathcal{C}$ is a functor that assigns to each object $C \in \mathcal{C}$, a subobject $\sigma(C)$ such that for each morphism $f : C \longrightarrow C'$ in $\mathcal{C}$, we have the commutative diagram*

$$\begin{array}{ccc} C & \xrightarrow{\ f\ } & C' \\ {\scriptstyle \iota}\uparrow & & \uparrow{\scriptstyle \iota} \\ \sigma(C) & \xrightarrow{\ \sigma(f)\ } & \sigma(C'). \end{array}$$

Here, $\sigma(f)$ is the restriction and corestriction of f to $\sigma(C)$ and $\sigma(C')$ respectively, that is, $\sigma(f) := f \mid_{\sigma(C)}: \sigma(C) \longrightarrow \sigma(C')$. Also, ι is represents the inclusion map.

Operations within objects of a category allow us to define operations between preradicals. This is the case when we consider the category of R-modules, where R is a commutative unitary ring, and the category of modular complete lattices $\mathcal{L}_{\mathcal{M}}$ (see [13]). For the purpose of this work, we will consider the category R-Mod, where we can define four principal operations: if $M \in R$-Mod and σ, τ are two preradicals, then

i) The meet $(\sigma \wedge \tau)(M) = \sigma(M) \cap \sigma(M)$;

ii) The join $(\sigma \vee \tau)(M) = \sigma(M) + \tau(M)$;

iii) The product $(\sigma \cdot \tau)(M) = \sigma(\tau(M))$;

(iv) The coproduct $(\sigma : \tau)(M)$ is such that

$$(\sigma : \tau)(M)/\sigma(M) = \tau(M/\sigma(M)).$$

[1] An endofunctor is a functor whose domain is equal to its codomein.

We will assume for now that the considered category is K-Mod with K a field, although the results in this sections holds for a commutative ring with unit R. A *diagram of shape* $\mathcal{J}$ in the category K-Mod is a functor $F : \mathcal{J} \longrightarrow K\text{-}Mod$. Commonly, we use *small* categories to define diagrams in a category, since these have only a set's worth of arrows (and thus, also of objects since every object in the category defines an identity morphisms). For instance, if we have a linear ordered shape diagram (that is, where J is a poset)

$$\cdots \longrightarrow M_i \xrightarrow{f} M_{i+1} \xrightarrow{g} \ldots \longrightarrow M_j \xrightarrow{h} M_{j+1} \longrightarrow \cdots$$

we can apply any preradical σ obtaining the commutative diagrams

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & M_i & \xrightarrow{f} & M_{i+1} & \xrightarrow{g} & \ldots & \longrightarrow & M_j & \xrightarrow{h} & M_{j+1} & \longrightarrow & \cdots \\
\uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} \\
\cdots & \longrightarrow & \sigma(M_i) & \xrightarrow{f_{|}} & \sigma(M_{i+1}) & \xrightarrow{g_{|}} & \ldots & \longrightarrow & \sigma(M_j) & \xrightarrow{h_{|}} & \sigma(M_{j+1}) & \longrightarrow & \cdots .
\end{array}$$

Since the diagrams are all commutative, the collection $\{\sigma(M_i)\}_{i\in J}$ tells us how the information, regarding to σ, flows through the diagram of shape J.

We will denote by K-pr the collection of all preradicals on K-Mod, where K is a field. There is a natural partial ordering in K-pr given by $\sigma \leq \tau$ if and only if $\sigma(M) \leq \tau(M)$. Also, as the intersection and sum of an arbitrary family of K-modules is also an K-module, then both, the join and the meet operations can be defined for any family of preradicals making K-pr a big lattice [2]: for any family $\{\tau_\alpha\}_{\alpha\in I}$ of preradicals and K-module M,

i) $\left(\bigvee_{\alpha\in I} \tau_\alpha\right)(M) = \Sigma_{\alpha\in I}\, \tau_\alpha(M)$,

ii) $\left(\bigwedge_{\alpha\in I} \tau_\alpha\right)(M) = \cap_{\alpha\in I}\, \tau_\alpha(M)$.

The four main operations defined on K-pr enable us to describe the information of the system in each component M:

- The meet of a family $\{\sigma\}_{i\in I}$ of preradicals gives us the minimum information present in σ_i, for each $i \in I$.
- The join of a family $\{\sigma\}_{i\in I}$ of preradicals gives us the maximum information generated by all σ_i, with $i \in I$.
- The product of $\sigma \cdot \tau$ gives us the information of σ after the information of τ is processed.
- The coproduct $(\sigma : \tau)$ gives the information of τ which is conditioned to contain the information of σ.

Each preradical σ can be considered as a compatible choice assignments on the category K-Mod in the following sense: for a preradical σ, the submodule $\sigma(C)$ of C is compatible with any other submodule $\sigma(D)$ of D, whenever there is a morphism $f : C \longrightarrow D$. Thus, if we regard that how information is transmitted within a category is through the collection of all the morphisms that belongs to the category, then these compatible choice assignments can be considered as how compatible information is transmitted within the category.

In K-Mod, we can define the following two preradicals. First, given a K-module M and a submodule N, the preradical α_N^M is such that evaluated in any K-module W is

[2]A big lattice is a class (not necessarily a set) having joins and meets for arbitrary families (indexed by a class) of elements.

$$\alpha_N^M(W) = \sum\{f(N) \mid f \in Hom(M, W)\},$$

where $Hom(M, W) = \{f : M \longrightarrow W)\}$. Second, given N, a submodule of M, one can define the preradical ω_N^M which evaluated in any K-module W is

$$\omega_N^M(W) = \cap\{f^{-1}(N) \mid f \in Hom(W, M)\},$$

where $Hom(W, M) = \{f : W \longrightarrow M)\}$ and $f^{-1}(N)$ denotes the inverse image of N under the morphism f. . One can easily prove that every preradical σ in K-Mod can be written as

$$\sigma = \bigvee\{\alpha^M_{\sigma(M)} | M \in K\text{-}Mod\} = \bigwedge\{\omega^M_{\sigma(M)} | M \in K\text{-}Mod\}.$$

We end this section with the next result which will be used in the discussions of Section 5:

Proposition 3.2. *[2][Proposition I.1.2] Let σ be a preradical and $\{M_i\}_{i\in I}$ be a family of R-modules. Then $\sigma(\oplus_{i\in I}\ M_i) = \oplus_{i\in I}\ \sigma(M_i)$.*

4 Persistence Through Preradicals

In this section we discuss our main technical contributions and results. First, observe how for any quiver Q one can define the *free category* or *path category* of the quiver Q, denoted as $\Lambda(Q)$, whose objects are the vertices of the quiver Q, and whose morphisms are paths between objects. Here, the composition operation is given by concatenation of paths. In this way, we can think of a G-module representation as a diagram of shape Q in K-Mod. This diagram is given by the functor $F : \Lambda(Q) \longrightarrow K\text{-}Mod$, which assigns to each object in Λ a K-vector space and to each arrow in $\Lambda(Q)$ a linear morphism. This fact will allow us to apply the α preradicals in K-Mod to express the flow of information within the G-module representation structure associated to a directed acyclic graph $G = (V, E)$. In this case, the morphisms that defines each α_N^M will be restricted to the ones that appear in the G-module representation.

Suppose now we have a single-source single-sink graph G and its commutative G-module representation $M = (\{W_v\}_{v\in V}, \{f_e\}_{e\in E})$

$$W_s \xrightarrow{f} W_i \xrightarrow{g} \dots \longrightarrow W_j \xrightarrow{h} W_t,$$

where W_s and W_t represents the source and the sink respectively. As its shown in [8, Lemma 3.3], the limit and colimit of the G-module M are W_s and W_t respectively. Thus, the persistence of M is

$$P(M) = im(W_s \xrightarrow{\varphi} W_t) = \varphi(W_s),$$

where φ is the linear function from the source W_s to the sink W_t. As we next show, the persistence $P(M)$ in [8, Lemma 3.3] can also be described using an α preradical:

Proposition 4.1. *Let $M = (\{W_v\}_{v\in V}, \{f_e\}_{e\in E})$ be a commutative G-module such that the underlying directed acyclic graph is a single-source single-sink graph, with vertices W_s and W_t respectively. The persistence shown in [8, Lemma 3.3], is the same as the preradical $\alpha^{W_s}_{W_s}$ evaluated in W_t. In this case, the considered morphisms $g : W_s \longrightarrow W_t$ are restricted to the ones appearing in the G-module representation.*

Proof. Let $M = (\{W_v\}_{v\in V}, \{f_e\}_{e\in E})$ be a commutative G-module such that the underlying directed acyclic graph is a single-source single-sink graph, with vertices W_s and W_t respectively. By definition, the preradical $\alpha_{W_s}^{W_s}$ evaluated at W_t is

$$\alpha_{W_s}^{W_s}(W_t) = \sum\{g(W_s) \mid g \in Hom(W_s, W_t)\}.$$

Here, $Hom(W_s, W_t)$ is taken as all the linear morphisms in K-Mod, from W_s to W_t. Now, if we restrict the morphisms in $Hom(W_s, W_t)$ to the ones that appear in the commutative G-module representation, we will still have the commutative diagrams

$$\begin{array}{ccccccccc}
W_s & \xrightarrow{f} & W_i & \xrightarrow{g} & \dots & \longrightarrow & W_j & \xrightarrow{h} & W_t \\
\uparrow{\scriptstyle\iota} & & \uparrow{\scriptstyle\iota} & & & & \uparrow{\scriptstyle\iota} & & \uparrow{\scriptstyle\iota} \\
\alpha_{W_s}^{W_s}(W_s) & \xrightarrow{f_|} & \alpha_{W_s}^{W_s}(W_i) & \xrightarrow{g_|} & \dots & \longrightarrow & \alpha_{W_s}^{W_s}(W_j) & \xrightarrow{h_|} & \alpha_{W_s}^{W_s}(W_t).
\end{array}$$

Since the G-module is commutative we have that $g(W_s) = \varphi(M)$ for every $g \in Hom(W_s, W_t)$. Thus,

$$\begin{aligned}
\alpha_{W_s}^{W_s}(W_t) &= \sum\{g(W_s) \mid g \in M \wedge g \in Hom(W_s, W_t)\} \\
&= \sum \varphi(M) = \varphi(M) = im(\varphi) = P(M).
\end{aligned}$$

□

From the preceding Proposition, we notice that if σ is any other preradical in K-Mod, we can apply it to each component of the diagram that comprise the G-module representation, obtaining commutative diagrams

$$\begin{array}{ccccccccc}
W_s & \xrightarrow{f} & W_i & \xrightarrow{g} & \dots & \longrightarrow & W_j & \xrightarrow{h} & W_t \\
\uparrow{\scriptstyle\iota} & & \uparrow{\scriptstyle\iota} & & & & \uparrow{\scriptstyle\iota} & & \uparrow{\scriptstyle\iota} \\
\sigma(W_s) & \xrightarrow{f_|} & \sigma(W_i) & \xrightarrow{g_|} & \dots & \longrightarrow & \sigma(W_j) & \xrightarrow{h_|} & \sigma(W_t).
\end{array}$$

Since each diagram commutes and the G-module M is commutative, it follows that the information defined by the preradical σ that persists through the G-module representation is $\varphi(\sigma(W_s)) \subset im(\varphi) \cap \sigma(W_t)$. Furthermore, we can apply any of the four operations $\sigma \wedge \tau$, $\sigma \vee \tau$, $\sigma \cdot \tau$ and $(\sigma : \tau)$ between any two preradicals σ and τ, to the G-module representation obtaining in each case different information. The fact that this information persists through the G-module representation is due to the commutative diagrams property which preserves the underlying structure.

As shown in [8, Proposition 3.4], DAG persistence generalizes the Standard persistence, the Zigzag persistence and the Multidimensional persistence. In their proof, the authors rely on [8, Lemma 3.3] to show that the Standard and Multidimensional persistence homology groups can be defined in terms of the image of a map, whereas in the case of zigzag persistence, the definition of the zigzag persistence module is identical to the definition of DAG persistence module. In the following result we show the corresponding proposition using preradicals.

Proposition 4.2. *Suppose χ_G is a graph filtration of X. Then:*

(1) (Standard Persistence)
If G is the graph corresponding to the filtration $X_0 \to X_1 \to \cdots \to X_n$ and $I_{i,p}$ is the subgraph consisting of vertices $\{X_i, \cdots, X_p\}$ then $H_k^{I_{i,p}}(\chi_G) \cong \alpha_{H_k(X_i)}^{H_k(X_i)}(H_k(X_{i+p}))$.

(2) *(Multidimensional Persistence)*
Let $\chi = \{X_v\}_{v\in\{0,\ldots,m\}^d}$ be a multifiltration with underlying graph G. If $G_{u,v}$ is the subgraph with vertices $\{w \in G | u \le w \le v\}$ then the rank invariant $\rho_{X,k}(u,v) = dim\ \alpha^{H_k(X_u)}_{H_k(X_u)}(H_k(X_v))$

Proof. (1) From the fact that $H_k^{I_{i,p}}(\chi_G) = im(H_k(X_i) \longrightarrow H_k(X_{i+p}))$ and that in a graph filtration the diagrams commute, it follows that

$$H_k^{I_{i,p}}(\chi_G) = im(H_k(X_i) \longrightarrow H_k(X_{i+p})) = \alpha^{H_k(X_i)}_{H_k(X_i)}(H_k(X_{i+p})).$$

(2) The rank invariant $\rho_{X,k}(u,v)$ is defined as the dimension of the image of the induced map $H_k(X_u) \longrightarrow H_k(X_v)$. Since the diagram commutes, any map which follows a path from u to v in the graph will have the same image. Thus, we have that

$$H_k^{G_{u,v}} = im(H_k(X_u) \longrightarrow H_k(X_v)) = \alpha^{H_k(X_u)}_{H_k(X_u)}(H_k(X_v)).$$

Hence, the rank invariant is the dimension of the subspace defined by the preradical

$$\alpha^{H_k(X_u)}_{H_k(X_u)}(H_k(X_v)).$$

□

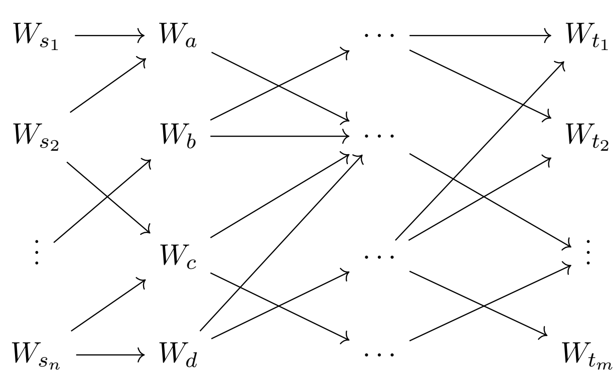
We now generalize Proposition 4.1 by assuming that the underlying DAG have n sources $s_1, \ldots, s_n$ and m sinks $t_1, \ldots, t_m$.

Proposition 4.3. *Let M be a G-module such that the underlying directed acyclic graph have n sources and m sinks. The persistence of the source W_{s_i} is obtained by evaluating the preradical*

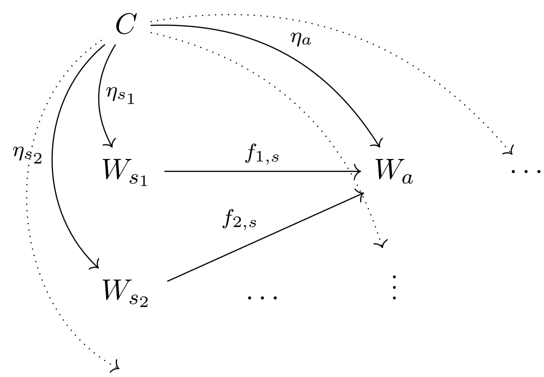
$$\alpha^{lim(M)}_{\oplus_{i=1}^n W_{s_i}}(colim(M)) = \sum\{f(\oplus_{i=1}^n W_{s_i}) \mid f : lim(M) \longrightarrow colim(M)\}$$

where the morphisms $f : lim(M) \longrightarrow colim(M)$ are restricted to the ones that appear in the extended G-module representation.

Proof. Let M be a commutative G-module such that the underlying directed acyclic graph have n sources and m sinks.



For any cone $(C, \{\eta_v\}_{v\in V})$ of M we have that all morphisms $\eta_j : C \longrightarrow W_j$ are determined by some $\eta_{s_i} : C \longrightarrow W_{s_i}$ with $i \in \{1, \ldots, n\}$ in the following way:

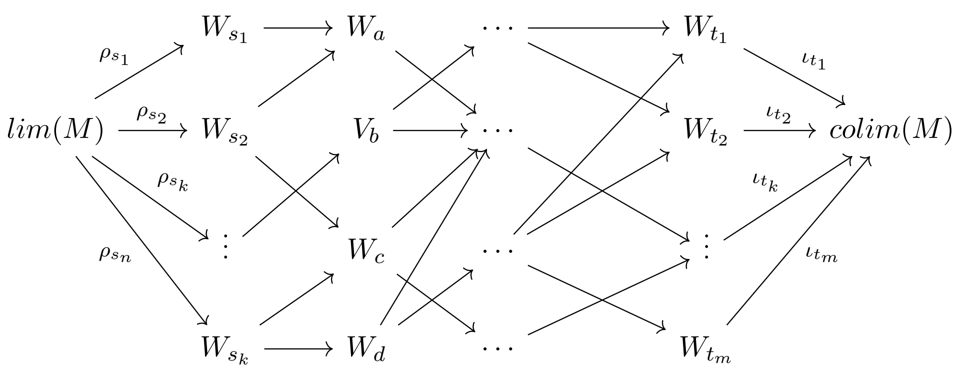


due to the commutativity property of a cone, we have that

$$f_{s_1,a} \circ \eta_{s_1} = \eta_a = f_{s_2,a} \circ \eta_{s_2}$$

that is, any η_l with $l \neq \{s_1, \cdots, s_n\}$ is factorized by some η_{s_i}, for $i = 1, \cdots, n$. Furthermore, the above argument shows that any two or more factorizations of η_l are equal. Thus, when considering the persistence of the commutative G-module M, it suffices to take the limit $lim(M)$ along with the morphisms $\rho_{s_i} : lim(M) \longrightarrow W_{s_i}$ for $i = 1, \cdots, n$. A similar argument shows that for the colimit, if suffices to take the morphisms $\iota_{t_i} : W_{t_i} \longrightarrow colim(M)$, for $i = 1, \cdots, m$.

By taking the limit and colimit of M we obtain an extension of the commutative G-module representation as a single-source single-sink graph:



Notice that the $lim(M) = \Pi_{j \in V}\ W_j$ which has W_{s_i} as a submodule for $i = 1, \ldots, n$ as well as the direct sum $\oplus_{i=1}^{n} W_{s_i}$. Hence, since all morphisms from $lim(M)$ to $colim(M)$ goes through some W_{s_i}, for $i = 1, \ldots, n$, if follows that the persistence of the commutative G-module M is the sum of the images of all morphisms from $lim(M)$ to $colim(M)$, that is,

$$\alpha^{lim(M)}_{\oplus_{i=1}^{n} W_{s_i}}(colim(M)) = \sum \{ f(\oplus_{i=1}^{n} W_{s_i}) \mid f : lim(M) \longrightarrow colim(M) \}$$

□

Using preradicals, we can also describe the persistence of information defined by a proper subset of sinks:

Corollary 4.4. *Let M be a commutative G-module such that the underlying directed acyclic graph have n sources and m sinks. The persistence of the sources $W_{s_{i_1}}, \cdots W_{s_{i_l}}$ is obtained by evaluating the preradical*

$$\alpha^{lim(M)}_{\oplus_{k=1}^{l} W_{s_{i_k}}}(colim(M)) = \sum\{f(\oplus_{k=1}^{l} W_{s_{i_k}}) \mid f : lim(M) \longrightarrow colim(M)\}$$

Critically, we note that these results can also be applied to any subspace L_{s_i} of W_{s_i} and hence, to any subspace of $\oplus_{k=1}^{l} W_{s_{i_k}}$. In such a case, by considering the preradical $\alpha^{lim(M)}_{L_{s_i}}$ we obtain the information from L_{s_i} that persists to $colim(M)$.

In a sense, the preradicals α^M_N fulfill the role of propagating information forward-wise. In other words, we obtain the *direct* information of N in K that comes from M (in this case morphisms goes from $M \longrightarrow K$). When the structure of the G-module allows having more morphisms in either direction between its components, we can make use of the ω preradicals to obtain information that also persists through the G-module representation. In this case, the direction of the morphisms considered in the ω preradicals is reversed, that is, now morphisms go from $K \longrightarrow M$ and we are taking the intersection of all inverse images of N. This *indirect* information persists through the G-module structure since for any linear morphisms $f : K \longrightarrow K'$ we obtain a commutative diagrams: given a vector space W_j and a subspace L_j, the preradical $\omega^{W_j}_{L_j}$ induces the following commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{f} & K' \\ \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} \\ \omega^{W_j}_{L_j}(K) & \xrightarrow{f_|} & \omega^{W_j}_{L_j}(K'). \end{array}$$

5 Preradicals and their Interpretation

In this last section we sketch how preradicals can be used to describe the flow of information from a general perspective. Motivated by principles from Shannon's version of information theory, where messages are created by choosing letters or words from a set, preradicals are also functions of choice whose choices are compatible with the structure of the category. This gives preradicals a high order approximation to the information that comprise the category itself.

Any system can be decomposed into a collection of objects called components, which are related to each other by an input-output relationship. These input-output relationships connect the components, and can be represented by an oriented arrow $A \longrightarrow B$ if an output of A is an input of B. This gives the system a directed graph representation. However, such a representation cannot describe many situations of the actual input-output existing relations between the components of the system. For instance, a component A may contribute to multiple distinct outputs, although the directed graph representation will only show one arrow connecting A to B. To overcome this, we say that two components A and B are connected if there exists a function $f : A \longrightarrow B$; this allows us to consider the set of all input-output relations between these two components. Thus, the set of functions describing the input-output relations is determined by the components A and B. We will denote this set by $Hom(A, B)$. It is well known that any directed graph can be associated with a category whose objects represent the vertices of the graph and whose morphisms correspond to the paths between vertices in the graph. For the purpose of this work, instead, we will consider the directed graph as a quiver, so that we can associate it with its G-module representation.

Let us consider a component B which receives two inputs from components A_1 and A_2. Then, we have two functions $f_1 : A_1 \longrightarrow B$ and $f_2 : A_2 \longrightarrow B$ which represents a possible output-input relationship between the respective source and target components. Both functions define a transformation from the direct sum (or direct product since it has a finite number of components) $f : A_1 \oplus A_2 \longrightarrow B$ such that $f \circ \iota_i = f_i$, that is,

$$\left(A_i \overset{\iota_i}{\hookrightarrow} A_1 \oplus A_2 \xrightarrow{f} B\right) = A_i \xrightarrow{f_i} B,$$

where $\iota_i : A_i \longrightarrow A_1 \oplus A_2$ denotes the inclusion map.

As σ is a preradical, then $\sigma(A_1 \oplus A_2) = \sigma(A_1) \oplus \sigma(A_2)$ and thus the diagram

$$\begin{array}{ccc} A_1 \oplus A_2 & \xrightarrow{f} & B \\ \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} \\ \sigma(A_1) \oplus \sigma(A_2) & \xrightarrow{f|} & \sigma(B) \end{array}$$

commutes. With this in mind, we obtain the information defined by the compatible choice assignment σ relative to the transformation f within the system. This turns out to be

$$f(\sigma(A_1)) \oplus f(\sigma(A_2)) \subseteq \sigma(B) \subseteq B.$$

We can generalize this argument to a number of n inputs from the components $A_1, \cdots, A_n$. In this case, we get a set of n functions

$$\begin{array}{c} A_1 \xrightarrow{f_1} B, \\ \vdots \\ A_n \xrightarrow{f_n} B, \end{array}$$

which induces a transformation $f : \oplus_{i=1}^n A_i \longrightarrow B$ such that $f \circ \iota_i = f_i$, that is,

$$\left(A_i \overset{\iota_i}{\hookrightarrow} \oplus_{i=1}^n A_i \xrightarrow{f} B\right) = A_i \xrightarrow{f_i} B, \text{ for each } i \in \{1, \cdots, n\}.$$

As σ is a preradical, then $\sigma(\oplus_i^n A_i) = \oplus_{i=1}^n \sigma(A_i)$ and thus

$$\begin{array}{ccc} \oplus_{i=1}^n A_i & \xrightarrow{f} & B \\ \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} \\ \oplus_{i=1}^n \sigma(A_i) & \xrightarrow{f|} & \sigma(B) \end{array}$$

is a commutative diagram. Therefore, the information defined by the compatible choice assignment σ relative with the transformation f is

$$f(\oplus_{i=1}^n \sigma(A_i)) = f(\sigma(A_1)) \oplus f(\sigma(A_2)) \oplus \cdots \oplus f(\sigma(A_n)) \subseteq \sigma(B) \subseteq B.$$

In the above construction, we obtained information about the system using a single preradical σ. We now show that the join operation in R-pr allows us to obtain information defined by a finite set of preradicals. In this case, each component will have an associated preradical which will give us information about the input-output relationship between the source and the target components. The whole construction will gather the amount of information generated by the set of preradicals relative to the functions that comprise the input-output relationships among sources and target components. Suppose we have two functions $f_1 : A_1 \longrightarrow B$ and $f_2 : A_2 \longrightarrow B$ which again represents a possible input-output relationship between the respective source and target components. As we saw above, these functions define a transformation from the direct sum (or product since it is a finite index)

$f : A_1 \oplus A_2 \longrightarrow B$ such that $f \circ \iota_i = f_i$ for $i = 1, 2$. If σ and τ are two preradicals, then we have the commutative diagrams

$$\begin{CD} A_1 @>f_1>> B \\ @A{\iota}AA @AA{\iota}A \\ \sigma(A_1) @>f_1|>> \sigma(B) \end{CD} \quad \text{and} \quad \begin{CD} A_2 @>f_2>> B \\ @A{\iota}AA @AA{\iota}A \\ \tau(A_2) @>f_2|>> \tau(B). \end{CD} \tag{*}$$

Since $f \circ \iota_i = f_i$ for $i = 1, 2$ and $\sigma(B) + \tau(B) \subseteq B$, from (*) we have that

$$\begin{CD} A_1 \oplus A_2 @>f>> B \\ @A{\iota}AA @AA{\iota}A \\ \sigma(A_1) \oplus \tau(A_2) @>f|>> \sigma(B) + \tau(B) \end{CD}$$

is also a commutative diagram. Notice that $\sigma(B) + \tau(B) = (\sigma \vee \tau)(B)$.

The last argument holds for a number of n inputs from components $A_1, \cdots, A_n$ to the component B in a analogous way. In this case, if $\sigma_1, \cdots \sigma_n$ are n preradicals, for each $i \in \{1, \cdots, n\}$ we have the commutative diagrams

$$\begin{CD} A_i @>f_i>> B \\ @A{\iota}AA @AA{\iota}A \\ \sigma_i(A_i) @>f_i|>> \sigma_i(B) \end{CD}$$

which imply the commutativity of the following diagram:

$$\begin{CD} \oplus_{i=1}^n A_i @>f>> B \\ @A{\iota}AA @AA{\iota}A \\ \oplus_{i=1}^n \sigma_i(A_i) @>f|>> \Sigma_{i=1}^n \sigma_i(B). \end{CD}$$

Observe that $\Sigma_{i=1}^n \sigma_i(B)$ is actually $(\sigma_1 \vee \cdots \vee \sigma_n)(B)$, so as we would expect, the information defined by the set of compatible choice assignments $\sigma_1, \cdots, \sigma_n$ relative to the transformation f is at most $(\sigma_1 \vee \cdots \vee \sigma_n)(B)$.

We end this section by illustrating the broader case when the system has multiple output-input relationships between two components. Suppose we have components $A_1, \cdots, A_n$ whose outputs are the input of a component B_1, which in turn is also connected to C_1 and C_2 (Figure 1).

To obtain the total information that component B_1 gets from components $A_1, \cdots, A_n$ we must take into consideration the limit $lim(A)$ of the objects $A_1, \cdots, A_n$ and define the preradical α using $lim(A)$. In this way, the total information obtained in component B_1 is

$$\alpha_{lim(A)}^{lim(A)}(B_1) = \sum\{\gamma(lim(A)) | \gamma : lim(A) \longrightarrow B_1\} = \sum_{i=1}^3 (f_i \circ \rho_1)(lim(A)) + \sum_{j=1}^2 (g_j \circ \rho_2)(lim(A))$$

$$+ \cdots + \sum_{r=1}^2 (h_r \circ \rho_1)(lim(A)) = \sum_{i=1}^3 f_i(A_1) + \sum_{j=1}^2 g_j(A_2) + \cdots + \sum_{r=1}^2 h_r(A_n).$$

Here, we notice that it suffices to take the limit of the objects $A_1, \cdots A_n$ and not of all the objects that comprise the system; this is because we are interested in the information that is obtained from just these sources. In case two components A_i and A_j, have any input-output relationship, the

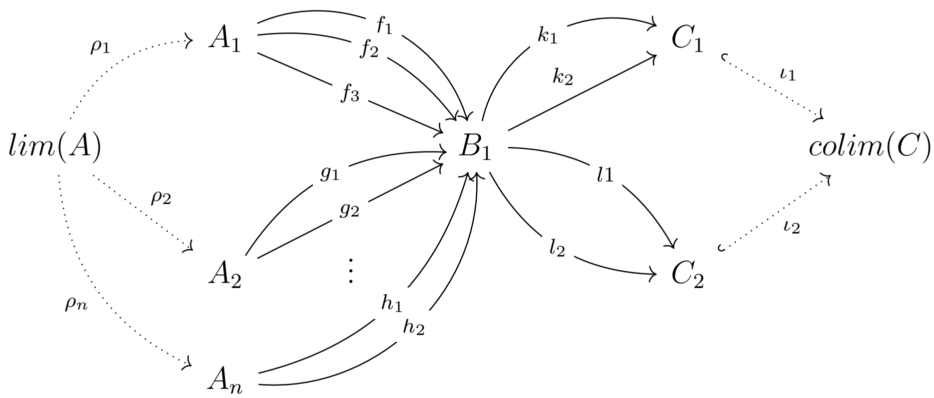


Figure 1: Components with multiple connections.

limit's commutativity property assure us that the ρ functions defined the limit construction also carry that information when considered in the above equation.

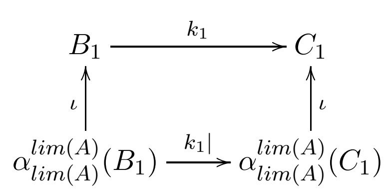
On the other hand, to get information from the collection of components A_i's to the component C_1, we take into account the preradical

$$\alpha^{lim(A)}_{lim(A)}(C_1) = \sum\{\gamma(lim(A))|\gamma : lim(A) \longrightarrow C_1\}$$
$$= \sum_{i=1}^{3}(k_1 \circ f_i)(A_1) + \sum_{j=1}^{2}(k_1 \circ g_j)(A_2) + \cdots + \sum_{r=1}^{2}(k_1 \circ h_r)(A_n)$$
$$+ \sum_{i=1}^{3}(k_2 \circ f_i)(A_1) + \sum_{j=1}^{2}(k_2 \circ g_j)(A_2) + \cdots + \sum_{r=1}^{2}(k_2 \circ h_r)(A_n)$$

Observe that the information obtained from the source components which is also provided by the morphism $k_1 : B_1 \longrightarrow C_1$, is

$$\sum_{i=1}^{3}(k_1 \circ f_i)(A_1) + \sum_{j=1}^{2}(k_1 \circ g_j)(A_2) + \cdots + \sum_{r=1}^{2}(k_1 \circ h_r)(A_n)$$
$$= k_1\Big(\sum_{i=1}^{3} f_i(A_1) + \sum_{j=1}^{2} g_j(A_2) + \cdots + \sum_{r=1}^{2} h_r(A_n)\Big),$$

which correspond to the image in the commutative diagram:



Any of the above arguments can be used to obtain the information from any subset $A_{i_1}, \cdots, A_{i_m}$ from the components $A_1, \cdots, A_n$ to any other component. In that case, we must consider the submodule $\oplus_{j=1}^{m} A_{i_j}$ of $lim(A)$ and apply the preradical $\alpha^{lim(A)}_{\oplus_{j=1}^{m} A_{i_j}}$. Finally, to obtain the information that persists from $lim(A)$ through the objects of the system, we must take into consideration the $colim(C)$ and apply the preradical $\alpha^{lim(A)}_{lim(A)}$ to all functions $f : lim(A) \longrightarrow colim(C)$ and then take the sum of their images, just as in Proposition 4.3; this gives us the total information from the components that flows through all the system.

6 Conclusions

Preradicals are a tool from category theory that naturally describes the flow of information through a system. By definition, these structures show how information is transmitted in such a way that the underlying structure of the modeled system is preserved. Particularly, when we use a G-module representation associated with a directed acyclic graph model, preradicals characterize the notion of persistence in a G-module structure. Also, by the way they are defined, each preradical can be considered as a compatible choice assignment that shows how certain substructures are related within the modeled system. From a communication theory perspective, this provides a high order approximation to the information that contained in the system.

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A Homology groups

Given a simplicial complex X, a k-chain is a formal sum $\Sigma_{i=1}^{n} c_i S_i$ where each c_i is an integer and S_i is an oriented k-simplex. The collection of k-chains forms a group which we denote by $C_k(X)$. In fact, $C_k(X)$ is a free abelian group whose basis is in one-to-one correspondence with the set of k-simplicies in X. There is a *boundary* homomorphisms

$$\delta_k : C_k(X) \longrightarrow C_{k-1}(X)$$

that calculates the boundary of a chain and which satisfies $\delta_{k-1} \circ \delta_k = 0$ for all k. The kernel of $\delta_k : C_k(X) \longrightarrow C_{k-1}(X)$ is called the *cycle group* and we denote it as $Z_k(X)$. The image of $\delta_{k+1} : C_{k+1}(X) \longrightarrow C_k(X)$ is called the *boundary group* and is denoted by B_k. Both Z_k and B_k are contained in $C_k(X)$, and since $\delta_{k-1} \circ \delta_k = 0$, they satisfy $B_k \subseteq H_k(X)$. Thus, the k-homology group of X is defined as $H_k(X) = Z_k/B_k$. When the coefficients in the formal sum are taken in a field F, then $C_k(X), Z_k(X), B_k(X)$ and $H_k(X)$ are all vector spaces.

Given a filtration $X_0 \subseteq X_1 \subseteq \cdots X_n$, the persistence homology group $H_k^p(X_j)$ is defined as the image of the homomorphism induced by the inclusion map $\iota : X_j \hookrightarrow X_{j+t}$. In other words, the persistence homology group $H_k^p(X_j) = Im(\iota^*)$ where $\iota^* : H_k(X_j \longrightarrow H_k(X_{j+t})$. Notice that $H_k^p(X_j)$ can be considered as a subgroup of $H_k^p(X_{j+t})$

B G-module representation and preradicals

In the theory of G-module representation, one important fact is that each G-module have a unique representation in indecomposibles, which are G-modules that cannot be written as a non-trivial connected sum. This representation is unique as stated in the Theorem for finite dimensional Algebras:

Theorem B.1. *(Krull-Remak-Schmidt)*
The decomposition of a commutative G-module is unique up to isomorphisms and permutation of the summands.

The decomposition into indecomposibles allows us to compare when two G-modules are the same up to isomorphisms. Since we are interested in finite commutative G-modules, then each representation have the form $V = V_1 \oplus \cdots \oplus V_n$, where each V_i is indecomposible. When modeling data, this finiteness property makes computation more efficient when comparing two representations.

There are also other categories which can be used to make quiver representation and that satisfies the property of decomposition into indecomposibles. Such is the case of finite dimensional algebras over an algebraically closed field F and representations over artinian integral domain rings.

On the other hand, having a finite set of preradicals can be helpful to cluster information when modeling data, in a efficient way. We now recall the definition of a ring R of having finite representation type, in order to show these conditions.

Definition B.2. *[10, Definition 3.1] A ring R is said to have finite representation type if it is left artinian and if there are, up to isomorphisms, finitely many indecomposible finitely generated left R-modules.*

It is well known that any integral domain and artinian ring R is a field. When this is the case, we have that the lattice of preradicals in R-Mod is finite, as is it stated in

Theorem B.3. *[10, Theorem 3.5] For a commutative ring R the following conditions are equivalent:*

(i) R is an artinian principal ideal ring.

(ii) R-pr is finite.

(iii) R-pr is an artinian (and/or noetherian) lattice.

(iv) R is a ring of finite representation type.

Hence, by using a representation with objects in a category R-Mod, where R is a artinian principal ideal ring, we will have a finite set of preradicals with which we can see the flow of information and the persistence through the representation. This, when applied to a system, can save running time.