Annals of Mathematics
Vol. 46, No. 4, October, 1945

A GENERALIZATION OF THE RELATIVISTIC THEORY OF GRAVITATION

By Albert Einstein

(Received June 19, 1945)

Every attempt to establish a unified field theory must start, in my opinion, from a group of transformations which is no less general than that of the continuous transformations of the four coordinates. For we should hardly be successful in looking for the subsequent enlargement of the group for a theory based on a narrower group. It is further reasonable to attempt the establishment of a unified theory by a generalization of the relativistic theory of gravitation. Such a generalization, which does not seem to have been discovered so far, is described in the following.

If we speak about a unified theory we have two possible points of view, whose distinction is essential for the following:

(1) That the field appear as a unified covariant entity. As an example I cite the unification of the electric and the magnetic fields by the special theory of relativity. The unification here consists in this that the entire field considered is described as a skew-symmetric tensor. The basic group of Lorentz transformations does not enable us to split this field independently of the system of coordinates, into an electric and a magnetic one.

(2) Neither the field equations nor the Hamiltonian function can be expressed as the sum of several invariant parts, but are formally unified entities. Also this (weaker) criterion of uniformity is satisfied in our example of the special relativistic description of Maxwell's equations.

The theory we shall describe is unified according to criterion (2), but not according to criterion (1). Such a theory is to be considered unified only in a limited sense.

Structure of field and group

The above field is described by a tensor g_{ik} with complex components. These components shall satisfy a condition of symmetry which constitutes the natural generalization of the condition of symmetry of the metric field of the theory of gravitation to the complex domain, which we call "Hermitian symmetry":

$$g_{ik} = \overline{g_{ki}}\,. \tag{1}$$

The components are continuous functions of the four real coordinates $x_1, \cdots, x_4$. From (1) follows that the g_{ik} split according to:

$$g_{ik} = s_{ik} + ia_{ik}\,,$$

where s_{ik} and a_{ik} satisfy the conditions:

$$s_{ik} = s_{ki}$$

$$a_{ik} = -a_{ki}\,.$$

The group shall be, as in the theory of gravitation, that of the real continuous coordinate transformations. Relative to this group s_{ik} and a_{ik} are independent tensors. The field is therefore not unified with respect to criterion (1). On the other hand, we shall see that criterion (2) can be satisfied in an extremely natural manner. In this, as well as in the close connection to the relativistic theory of gravitation, I see the formal justification for the following field theory.

To the covariant tensor g_{ik} we can associate a contravariant one g^{ik} uniquely by the condition:

(2) $$g_{ki}g^{kl} = g_{ik}g^{lk} = \delta_i^l$$

where δ_i^l is the Kronecker tensor. Because of (1) the determinant:

$$g = | g_{ik} |$$

is real. For $\bar{g} = | \overline{g_{ik}} | = | g_{ki} | = | g_{ik} |$. We choose as in the theory of gravitation: $g < 0$ because of the special character of the timelike dimension.

The infinitesimal parallel translation

We now introduce a complex quantity $\Gamma_i{}^l{}_k$ which transforms like the corresponding quantities in Riemannian geometry. In analogy to the corresponding quantities of Riemannian geometry, the $\Gamma_i{}^l{}_k$ shall be Hermitian symmetric with respect to the lower indices.

(3) $$\Gamma_i{}^l{}_k = \overline{\Gamma_k{}^l{}_i}.$$

REMARKS: $\Gamma_i{}^l{}_k - \Gamma_k{}^l{}_i$ is a (purely imaginary) tensor. $\Gamma_k{}^l{}_i$ has the same law of transformation as $\Gamma_i{}^l{}_k$ (the same holds for $\frac{1}{2}(\Gamma_i{}^l{}_k + \Gamma_k{}^l{}_i)$). By contraction of the tensor $\frac{1}{2}(\Gamma_i{}^l{}_k - \Gamma_k{}^l{}_i)$ we get the vector:

(4) $$\Gamma_i = \tfrac{1}{2}(\Gamma_i{}^b{}_b - \Gamma_b{}^b{}_i).$$

From the fundamental laws it follows that here the parallel translation of a complex vector is not a unique operation for given Γ. We therefore introduce the following symbols in order to remove this indetermination:

(5) $$\begin{cases} \delta A^{\overset{i}{+}} = -\Gamma_s{}^i{}_t A^s\, dx_t \\ \delta A^{\overset{i}{-}} = -\Gamma_t{}^i{}_s A^s\, dx_t \\ \delta A^{\overset{i}{0}} = -\tfrac{1}{2}(\Gamma_s{}^i{}_t + \Gamma_t{}^i{}_s)A^s\, dx_t\,. \end{cases}$$

Corresponding symbols are introduced for the infinitesimal parallel translation of covariant tensors as well as for covariant differentiation: e.g.:

$$A^{\overset{i}{+}}{}_{;k} = A^i{}_{,k} + A^s\Gamma_s{}^i{}_k$$

$$A_{\underset{-}{i};k} = A_{i,k} - A_s\Gamma_k{}^s{}_i \qquad \text{etc.}$$

We now have to determine the Γ belonging to a given g_{ik} field by definition. We set:

(6) $$0 = g_{ik;l} = g_{\underset{+}{i}\underset{-}{k};l} = g_{ik,l} - g_{sk}\Gamma_i{}^s{}_l - g_{is}\Gamma_l{}^s{}_k\,.$$

For the following it is essential to realize that the right side of Eq. (6) has tensor character even if Eq. (6) is not satisfied. For the Kronecker tensor we get:

$$\delta_{\underset{+}{i}}{}^{\overset{k}{+}}{}_{;l} = \delta_i{}^s \Gamma_s{}^k{}_l - \delta_s{}^k \Gamma_i{}^s{}_l = 0 = \delta_{\underset{-}{i}}{}^{\overset{k}{-}}{}_{;l} = \delta_{\underset{0}{i}}{}^{\overset{k}{0}}{}_{;l} .$$

On the other hand:

$$\delta_{\underset{-}{i}}{}^{\overset{k}{+}}{}_{;l} = \Gamma_i{}^k{}_l - \Gamma_l{}^k{}_i = -\delta_{\underset{+}{i}}{}^{\overset{k}{-}}{}_{;l} \neq 0.$$

Therefore, in contracting tensors under the symbol of differentiation one has to watch the character of the indices carefully. Only for indices of the same character are the operations of contraction and of absolute differentiation interchangeable.

The special choice of the symmetry property of Γ and the differentiation in Eq. (6) is justified by the following. If one forms the Hermitian conjugate of the right-hand side of Eq. (6), i.e. if one interchanges i and k and then passes to the conjugate complex, one gets:

$$\overline{g_{ki,l}} - \overline{g_{si}}\,\overline{\Gamma_k{}^s{}_l} - \overline{g_{ks}}\,\overline{\Gamma_l{}^s{}_i}$$

or

$$g_{ik,l} - g_{is}\Gamma_l{}^s{}_k - g_{sk}\Gamma_i{}^s{}_l ,$$

i.e. Eq. (6) (or rather the right-hand side) coincides with its Hermitian conjugate form. This is necessary in order that for a given field the Γ be determined (but not over-determined).

If one multiplies (6) by g^{ik} and contracts, one gets, considering (2):

$$\frac{g_{,l}}{g} - (\Gamma_i{}^i{}_l + \Gamma_l{}^i{}_i) = 0. \tag{7}$$

The left-hand side of (7) has vector character independent of whether Eq. (7) is satisfied or not.

If we multiply Eq. (6) by $-g^{it}g^{sk}$ and sum with respect to i and k, then, since we have, because of (2):

$$g^{it}g_{ik,l} + g^{it}{}_{,l}g_{ik} = 0$$

we get

$$0 = g^{st}{}_{,l} + g^{bt}\Gamma_b{}^s{}_l + g^{sb}\Gamma_l{}^t{}_b = g^{\overset{s}{+}\overset{t}{-}}{}_{;l} . \tag{6a}$$

The main difference of the generalized theory as compared to the pure theory of gravitation, with regard to the equations determining Γ, lies in the fact that the equations which determine Γ in terms of the g-field cannot be solved in a simple manner.

Next to the concept of tensor, that of tensor-density is of importance. If e.g. A^i is a tensor (1st rank), then

$$\mathfrak{A}^i = \sqrt{-g}\,A^i$$

is the corresponding tensor-density, whereby the law of transformation is determined.

By differentiation we get, e.g.

$$A^{\underset{+}{i}}{}_{;k} = A^{i}{}_{,k} + A^{a}\Gamma_{a}{}^{i}{}_{k}\,.$$

If we multiply this by $\sqrt{-g}$ we get that

$$(\mathfrak{A}^{\underset{+}{i}}{}_{;k}) \equiv \mathfrak{A}^{i}{}_{,k} + \mathfrak{A}^{a}\Gamma_{a}{}^{i}{}_{k} - \tfrac{1}{2}\mathfrak{A}^{i}\frac{g_{,k}}{g} \tag{6b}$$

is a tensor density, which we define to be the absolute derivative $\mathfrak{A}^{\underset{+}{i}}{}_{;k}$ of $\mathfrak{A}^{i}$. The last term takes account of the density character of $\mathfrak{A}^{i}$. The analogous fact holds for the differentiation of all tensor densities. In particular for a scalar density $\mathfrak{r}$ we have:

$$\mathfrak{r}_{;l} = \mathfrak{r}_{,l} - \tfrac{1}{2}\mathfrak{r}\frac{g_{,l}}{g}\,.$$

If we set $\mathfrak{r} = \sqrt{-g}$ then the absolute derivative vanishes.

If we set $\mathfrak{g}^{ik} = \sqrt{-g}\,g^{ik}$ then

$$\mathfrak{g}^{\underset{+}{i}\underset{-}{k}}{}_{;l} = \mathfrak{g}^{ik}{}_{,l} + \mathfrak{g}^{ak}\Gamma_{a}{}^{i}{}_{k} + \mathfrak{g}^{ia}\Gamma_{l}{}^{k}{}_{a} - \tfrac{1}{2}\mathfrak{g}^{ik}\frac{g_{,l}}{g} = 0 \tag{6c}$$

where (6c) follows from (6b) and from the definition of tensor density.

Curvature

We start from the expression for parallel translation, e.g. according to the first of the equations (5). By translation of a complex vector along the boundary of an infinitesimal (plane) surface-element, one obtains a (complex) tensor of curvature just as in the theory of real fields.

One thus obtains the complex curvature-tensor

$$\Gamma_{kl}{}^{i}{}_{,m} - \Gamma_{km}{}^{i}{}_{,l} - \Gamma_{a}{}^{i}{}_{l}\Gamma_{k}{}^{a}{}_{m} + \Gamma_{a}{}^{i}{}_{m}\Gamma_{k}{}^{a}{}_{l}\,. \tag{8}$$

Contracting this according to the indices i and m we get the tensor

$$\Gamma_{kl}{}^{a}{}_{,a} - \Gamma_{k}{}^{a}{}_{b}\Gamma_{a}{}^{b}{}_{l} - \Gamma_{ka}{}^{a}{}_{,l} + \Gamma_{k}{}^{a}{}_{l}\Gamma_{a}{}^{b}{}_{b}\,. \tag{9}$$

By taking the mean value of this tensor and its Hermitian conjugate we get the Hermitian tensor

$$R_{ik} = \Gamma_{ik}{}^{a}{}_{,a} - \Gamma_{i}{}^{a}{}_{b}\Gamma_{a}{}^{b}{}_{k} - \tfrac{1}{2}(\Gamma_{ia}{}^{a}{}_{,k} + \Gamma_{ak}{}^{a}{}_{,i}) + \tfrac{1}{2}\Gamma_{i}{}^{a}{}_{k}(\Gamma_{a}{}^{b}{}_{b} + \Gamma_{b}{}^{b}{}_{a}). \tag{10}$$

Derivation of the field equations

It is now our aim to determine field equations which are compatible with our definitions (6). This we achieve through the application of a method which is already known from the theory of gravitation. For the time being we introduce g_{ik} and $\Gamma_{i}{}^{l}{}_{k}$ as independent field quantities, without assuming that they are combined by Eq. (6). From these quantities and their derivatives we construct a Hamiltonian density-function $\mathfrak{H}$ whose integral we vary independently with respect to the g and the Γ. $\mathfrak{H}$ is to be chosen so that the variation with respect to the Γ yields the Eq. (6). The variation with respect to g will then yield the proper field equations.

Derivation of the Hamiltonian function

We first construct a new tensor by subtracting a certain tensor S_{ik} from R_{ik}. According to (7) we have that

$$S_i = \frac{\partial \log \sqrt{-g}}{\partial x_i} - \tfrac{1}{2}(\Gamma_{i\ a}^{\ a} + \Gamma_{a\ i}^{\ a})$$

is a vector. From it we construct the tensor $S_{\underset{+}{i};k}$ $(= S_{ik})$ getting

$$\begin{aligned}(11)\qquad S_{ik} = {} & [(\log \sqrt{-g})_{,i,k} - (\log \sqrt{-g})_{,a}\Gamma_{i\ k}^{\ a}] \\ & - [\tfrac{1}{2}(\Gamma_{i\ a}^{\ a} + \Gamma_{a\ i}^{\ a})_{,k} - \tfrac{1}{2}(\Gamma_{a\ b}^{\ b} + \Gamma_{b\ a}^{\ b})\Gamma_{i\ k}^{\ a}].\end{aligned}$$

We get

$$(12)\qquad R_i{}^*{}_k = R_{ik} - S_{ik} = \Gamma_{i\ k,a}^{\ a} - \Gamma_{i\ b}^{\ a}\Gamma_{a\ k}^{\ b} - \log \sqrt{-g})_{,i,k} + (\log \sqrt{-g})_{,i,k}\Gamma_{i\ k}^{\ a}.$$

From this we construct with the help of the tensor-density $\mathfrak{g}^{ik}$ the Hamiltonian density-function

$$(13)\qquad \mathfrak{H} = R_{ik}{}^*\mathfrak{g}^{ik}.$$

The field equations

The variation of the integral of $\mathfrak{H}$ with respect to $\Gamma_{i\ k}^{\ a}$ and $\mathfrak{g}^{ik}$ yields after partial integration:

$$(14)\qquad \delta \int \mathfrak{H}\, d\tau = \int [(-\mathfrak{U}_a^{\ ik})\delta\Gamma_{i\ k}^{\ a} + G_{ik}\,\delta\mathfrak{g}^{ik}]\, d\tau.$$

For $\mathfrak{U}_a^{\ ik}$ we get, because of (12) and (13):

$$(14a)\qquad \mathfrak{U}_{a\ .}^{\ ik} = \mathfrak{g}^{ik}_{\ .\ ,a} + \mathfrak{g}^{bk}_{\ .}\Gamma_{b\ a}^{\ i} + \mathfrak{g}^{ib}\Gamma_{a\ .b}^{\ k} - \tfrac{1}{2}\mathfrak{g}^{ik}_{\ .}\frac{g_{,a}}{g} = \mathfrak{g}^{ik}_{\ .\ ;a}.$$

The variation with respect to $\mathfrak{g}^{ik}$ yields first the integrand

$$R_{ik}{}^*\,\delta\mathfrak{g}^{ik}_{\ .} - [\mathfrak{g}^{\ rs}_{,\ \ r,s} + (\Gamma_{r\ s}^{\ a}\mathfrak{g}^{rs})_{,a}]\delta(\log \sqrt{-g})$$

where

$$\frac{\delta g}{g} \equiv g^{ik}\,\delta g_{ik} \equiv \frac{1}{\sqrt{-g}}\,\mathfrak{g}^{ik}\,\delta g_{ik} \equiv \frac{1}{\sqrt{-g}}\,[\delta(4\sqrt{-g}) - g_{ik}\,\delta g^{ik}]$$

or

$$\frac{\delta g}{g} \equiv \frac{1}{\sqrt{-g}}\,g_{ik}\,\delta\mathfrak{g}^{ik} \equiv 2\delta(\log \sqrt{-g}).$$

Substituting this expression, we get as the result of variation with respect to the g

$$(14b)\qquad G_{ik} \equiv R_{ik}{}^* - \frac{1}{2\sqrt{-g}}\,[\mathfrak{g}^{\ rs}_{,\ \ r,s} + (\Gamma_{r\ s}^{\ a}\mathfrak{g}^{rs})_{,a}]g_{ik}.$$

The field equations following from our variation principle are then

$$(15)\qquad \begin{cases} \mathfrak{U}^{i\ k}_{\ a} = 0 \\ G_{ik} = 0. \end{cases}$$

The first system is equivalent to (6). The second system can be transformed, using the first. Namely from (6) we have

$$\mathfrak{g}^{as}{}_{;s} = 0 = \mathfrak{g}^{as}{}_{,s} + \mathfrak{g}^{bs}\Gamma_{b\ s}^{\ a} + \mathfrak{g}^{ab}\Gamma_{s\ b}^{\ s} - \tfrac{1}{2}\mathfrak{g}^{as}\frac{g_{,s}}{g}$$

or, because of (7)

$$\mathfrak{g}^{as}{}_{,s} + \mathfrak{g}^{bs}\Gamma_{b\ s}^{\ a} - \mathfrak{g}^{ab}\Gamma_b = 0 \quad \text{(where } \Gamma_b \text{ is as in equation (4).)}$$

In the same way, from

$$\mathfrak{g}^{sa}{}_{;a} = 0 = \mathfrak{g}^{sa}{}_{,s} + \mathfrak{g}^{ba}\Gamma_{b\ s}^{\ s} + \mathfrak{g}^{sb}\Gamma_{s\ b}^{\ a} - \tfrac{1}{2}\mathfrak{g}^{sa}\frac{g_{,s}}{g}$$

follows:

$$\mathfrak{g}^{sa}{}_{,s} + \mathfrak{g}^{sb}\Gamma_{s\ b}^{\ a} + \mathfrak{g}^{ba}\Gamma_b = 0.$$

We have therefore,

$$(16)\qquad \mathfrak{g}^{rs}{}_{,r,s} + (\Gamma_{r\ s}^{\ a}\mathfrak{g}^{rs})_{,a} = (\mathfrak{g}^{ab}\Gamma_b)_{,a} = -(\mathfrak{g}^{ba}\Gamma_b)_{,a} = (\mathfrak{g}^{ab}_{\vee}\Gamma_b)_{,a}$$

where $\mathfrak{g}^{ab}_{\vee}$ stands for the antisymmetric (imaginary) part of $\mathfrak{g}^{ab}$. We can therefore write

$$(14c)\qquad G_{ik} = R_{ik}^{*} - \frac{1}{2\sqrt{-g}}(\mathfrak{g}^{ab}\Gamma_b)_{,a}\,g_{ik}\,.$$

The field equations are therefore written explicitly

$$(15b)\qquad \begin{cases} 0 = g_{\underset{+-}{ik};l} = g_{ik,l} - g_{ak}\Gamma_{i\ l}^{\ a} - g_{ia}\Gamma_{l\ k}^{\ a} \\ 0 = G_{ik} = \Gamma_{i\ k,a}^{\ a} - \Gamma_{i\ b}^{\ a}\Gamma_{a\ k}^{\ b} - (\log\sqrt{-g})_{,i,k} \\ \qquad\qquad + (\log\sqrt{-g})_{,a}\Gamma_{i\ k}^{\ a} - \dfrac{1}{2\sqrt{-g}}(\mathfrak{g}^{ab}\Gamma_b)_{,a}\,g_{ik}\,. \end{cases}$$

The last term of G_{ik} vanishes in case of a real field. The remainder is then identical with the once contracted curvature tensor.

The equations are compatible since they are derived from a Hamiltonian principle; they are connected by a (real) quadruple-identity which can be derived according to a well known method.

The question whether these equations have physical significance is difficult to answer. One will tend to consider the antisymmetric part of the g_{ik} as a representation of the electromagnetic field, at least for infinitely small fields. However, the construction of the equations of first approximation show that they are weaker than the Maxwell equations. The fields can only be sufficiently characterized by the requirement that they be regular throughout the entire space. The physical test depends on the construction of exact solutions (if there are regular ones). This is a difficult task. However, the theory appears to be so natural as to justify great exertions.

ADDED IN PROOF:

The consideration of the field equations obtained suggests the adjunction of the four equations

$$\gamma_s = 0.$$

This is certainly permissible if there exist four additional identities between these equations and the field equations derived from the Hamiltonian in (15b). In that case the once contracted curvature (9) has Hermitian symmetry, the last member of the second equation (15b) vanishes and the right-hand side of these field equations becomes identical with the once contracted curvature (9).

In fact I have succeeded in establishing the above-mentioned identities. The clarification of this point will, however, be left for a separate paper, since the natural conception of the situation is connected with a new method for the derivation of the field equations.

THE INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N. J.