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ON A STATIONARY SYSTEM WITH SPHERICAL SYMMETRY CONSISTING OF MANY GRAVITATING MASSES

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If one considers Schwarzschild's solution of the static gravitational field of spherical symmetry

$$ds^2 = -\left(1 + \frac{\mu}{2r}\right)^4 (dx_1^2 + dx_2^2 + dx_3^2) + \left(\frac{1 - \frac{\mu}{2r}}{1 + \frac{\mu}{2r}}\right)^2 dt^2 \tag{1}$$

it is noted that

$$g_{44} = \left(\frac{1 - \frac{\mu}{2r}}{1 + \frac{\mu}{2r}}\right)^2$$

vanishes for $r = \mu/2$. This means that a clock kept at this place would go at the rate zero. Further it is easy to show that both light rays and material particles take an infinitely long time (measured in "coördinate time") in order to reach the point $r = \mu/2$ when originating from a point $r > \mu/2$. In this sense the sphere $r = \mu/2$ constitutes a place where the field is singular. (μ represents the gravitating mass.)

There arises the question whether it is possible to build up a field containing such singularities with the help of actual gravitating masses, or whether such regions with vanishing g_{44} do not exist in cases which have physical reality. Schwarzschild himself investigated the gravitational field which is produced by an incompressible liquid. He found that in this case, too, there appears a region with vanishing g_{44} if only, with given density of the liquid, the radius of the field-producing sphere is chosen large enough.

This argument, however, is not convincing; the concept of an incompressible liquid is not compatible with relativity theory as elastic waves would have to travel with infinite velocity. It would be necessary, therefore, to introduce a compressible liquid whose equation of state excludes the possibility of sound signals with a speed in excess of the velocity of light. But the treatment of any such problem would be quite involved; besides, the choice of such an equation of state would be arbitrary within wide limits, and one could not be sure that thereby no assumptions have been made which contain physical impossibilities.

One is thus led to ask whether matter cannot be introduced in such a way that questionable assumptions are excluded from the very beginning. In fact this can be done by choosing, as the field-producing mass, a great number of small gravitating particles which move freely under the influence of the field produced by all of them together. This is a system resembling a spherical star cluster. Hereby we may proceed as if the field, in which the particles are moving, were produced by a continuous mass distribution of spherical symmetry, corresponding to the whole of the particles.

We can further simplify our considerations by the special assumption that all particles move along circular paths around the center of symmetry of the cluster. Even in this case it is still possible to choose arbitrarily the radial distribution of mass density. The result of the following consideration will be that it is impossible to make g_{44} zero anywhere, and that the total gravitating mass which may be produced by distributing particles within a given radius, always remains below a certain bound.

1. On the paths of the particles and their spacial distribution

By a suitable choice of the radial coördinate, it is possible to obtain the gravitational field of the cluster of spherical symmetry in the form

$$ds^2 = -a(dx_1^2 + dx_2^2 + dx_3^2) + b\,dt^2, \tag{2}$$

whereby a and b are functions of $r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$. First we shall investigate the circular motion of one particle around the center of symmetry. Suppose, for instance, this motion takes place within the plane $x_3 = 0$. Through the introduction of polar coördinates

$$x_3 = r \cos \vartheta,$$

$$x_1 = r \sin \vartheta \cos \varphi,$$

$$x_2 = r \sin \vartheta \sin \varphi,$$

(2) assumes the form

$$ds^2 = -a[dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta\, d\varphi^2)] + b\,dt^2. \tag{2a}$$

The field is characterized by

$$g_{11} = -a, \qquad g_{33} = -ar^2 \sin^2 \vartheta,$$

$$g_{22} = -ar^2, \qquad g_{44} = b,$$

where all the rest of the $g_{\mu\nu}$ vanish. The particle under consideration satisfies the equation

$$\frac{d^2 x_\nu}{ds^2} + \Gamma^\nu_{\alpha\beta} \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} = 0. \tag{3}$$

In addition its motion is determined by the conditions

$$\frac{dx_1}{ds} = \frac{dr}{ds} = 0, \qquad \frac{d^2 x_3}{ds^2} = \frac{d^2 \varphi}{ds^2} = 0,$$

$$x_2 = \vartheta = \frac{\pi}{2}, \qquad \frac{d^2 x_4}{ds^2} = \frac{d^2 t}{ds^2} = 0.$$

It turns out that (3) is satisfied when

$$\Gamma^1_{33}\frac{dx_3}{ds}\frac{dx_3}{ds} + \Gamma^1_{44}\frac{dx_4}{ds}\frac{dx_4}{ds} = 0,$$

or when

$$-(ar^2)'\left(\frac{d\varphi}{dt}\right)^2 + b = 0. \tag{4}$$

Because of (2a), we have

$$\left(\frac{ds}{dt}\right)^2 = -ar^2\left(\frac{d\varphi}{dt}\right)^2 + b. \tag{5}$$

Thus, $d\varphi/dt$ and ds/dt are determined when the field is given.

Because ds^2 has to be positive for the world line of a particle in motion we have

$$\left(\frac{ds}{dt}\right)^2 = b - ar^2\left(\frac{d\varphi}{dt}\right)^2 = b - ar^2\frac{b'}{(ar^2)'} > 0,$$

or

$$1 - \frac{\frac{b'}{b}}{\frac{(ar^2)'}{ar^2}} > 0. \tag{6}$$

By applying this condition to Schwarzschild's field (1) we obtain

$$r > \frac{\mu}{2}(2 + \sqrt{3}). \tag{6a}$$

It follows that in the case of a Schwarzschild field a particle is bound to follow a path with a radius greater than $(2 + \sqrt{3})$ times the radius of the Schwarzschild singularity. This fact has the greatest significance for the following investigation: In the outermost layer of our particle cluster (and beyond it) the gravitational field is given by (1). It follows that the total gravitating mass of the cluster determines a lower limit for the radius of the cluster; this radius is (in coördinate measure) more than $(2 + \sqrt{3})$ times greater than the radius of the Schwarzschild singularity as defined by the field in the empty space outside the cluster.

The normal to the plane in which the particle considered moves has the direction of x_3. If it is assumed that the normals to an infinite number of such planes are distributed at random and also that the phase angles of the paths are subject to a random distribution, then we obtain a cluster of particles of spherical symmetry whose paths have the radius r. The most general cluster to be considered by us consists of an infinite number of clusters of this special type which belong to all values of r. (More accurately speaking, the whole cluster consists, of course, of a finite number of particles so that a field is created which only approximates spherical symmetry.)

In order to formulate the conditions of dynamical equilibrium of the cluster under the influence of its own gravitational field, we first have to compute the energy tensor belonging to such a cluster. For this purpose we assume, for the sake of simplicity that all particles have the same mass m.

2. The Matter-Energy Tensor of the Cluster

We consider the motion of particles within a volume element on the x_3-axis. The velocity vectors all have the same amount, they are perpendicular on the x_3-direction, and they are evenly distributed with respect to the directions within the x_1, x_2 - plane. We know further that the matter-energy tensor depends also on the particle density and on the gravitational potentials, but not on the derivatives of the latter. It is, therefore, possible to determine this tensor by a straightforward calculation.

First we consider particles, with the mass m and the particle density n_0 per unit volume, at rest with respect to a coördinate system of the theory of restricted relativity. In such a case of the energy tensor only the (44)-component exists,

$$T^{44} = mn_0 \frac{dx_4}{ds}\frac{dx_4}{ds}.$$

With respect to coördinate systems in relative motion in the x_1-direction we have the components

$$T^{11} = mn_0 \frac{dx_1}{ds}\frac{dx_1}{ds}, \qquad T^{44} = mn_0 \frac{dx_4}{ds}\frac{dx_4}{ds},$$

$$T^{14} = mn_0 \frac{dx_1}{ds}\frac{dx_4}{ds}.$$

The particle density n with respect to such a system is determined by the equations:

$$n_0 V_0 = nV, \qquad V_0\, ds = V\, dt,$$

where V_0 and V denote the rest volume and the coördinate volume respectively. Therefore we have

$$n_0 = n\frac{ds}{dx_4}.$$

We now consider the case when the velocity vector of the particle makes an angle α with respect to the x_1-axis, and is perpendicular to the x_3-axis. By using the relations derived above and by introducing $dl^2 = dx_1^2 + dx_2^2$, we obtain

$$T^{11} = mn\frac{ds}{dx_4}\left(\frac{dl}{ds}\right)^2 \cos^2\alpha, \qquad T^{12} = mn\frac{ds}{dx_4}\left(\frac{dl}{ds}\right)^2 \cos\alpha\sin\alpha,$$

$$T^{22} = mn\frac{ds}{dx_4}\left(\frac{dl}{ds}\right)^2 \sin^2\alpha, \qquad T^{14} = mn\frac{ds}{dx_4}\frac{dl}{ds}\frac{dx_4}{ds}\cos\alpha,$$

$$T^{44} = mn\frac{ds}{dx_4}\left(\frac{dx_4}{ds}\right)^2, \qquad T^{24} = mn\frac{ds}{dx_4}\frac{dl}{ds}\frac{dx_4}{ds}\sin\alpha,$$

all the other components of the energy tensor being zero. In the case that the velocity vectors are evenly distributed over all values of α the result is

$$T^{11} = T^{22} = \tfrac{1}{2}mn \frac{ds}{dx_4}\left(\frac{dl}{ds}\right)^2 = T_{11} = T_{22},$$

$$T^{44} = mn\frac{dx_4}{ds} = T_{44}.$$

We now proceed to the case that the components of the metric tensor are $g_{11} = g_{22} = g_{33} = -a$ and $g_{44} = b$. The components of the energy tensor are obtained by applying the transformation law for tensors and by transforming the coördinates according to

$$dx_a = a^{\frac{1}{2}}\,d\bar{x}_a$$

$$dx_4 = b^{\frac{1}{2}}\,d\bar{x}_4\,.$$

We obtain

$$\bar{T}_{11} = \left(\frac{dx_a}{d\bar{x}_a}\right)^2 T_{11} = aT_{11},$$

$$\bar{T}_{44} = \left(\frac{dx_4}{d\bar{x}_4}\right)^2 T_{44} = bT_{44}.$$

dl and dx_4, contained in T_{11} and T_{44}, are to be replaced dl by $a^{\frac{1}{2}}dl$ and dx_4 by $b^{\frac{1}{2}}d\bar{x}_4$. Further we have to introduce the particle density with respect to the new coördinates, $\bar{n}$, according to

$$n\,dx_1\,dx_2\,dx_3 = \bar{n}\,d\bar{x}_1\,d\bar{x}_2\,d\bar{x}_3$$

or

$$n = \bar{n}a^{-\frac{3}{2}}.$$

After having made all these transformations and substitutions, and omitting the bars denoting the new coördinate system, we obtain

$$\text{(7)}\quad \begin{cases} T_{11} = T_{22} = \tfrac{1}{2}mna^{\frac{1}{2}}b^{-\frac{1}{2}}\dfrac{ds}{dx_4}\left(\dfrac{dl}{ds}\right)^2, \\[2ex] T_{44} = mna^{-\frac{3}{2}}b^{\frac{3}{2}}\dfrac{dx_4}{ds}. \end{cases}$$

In these equations ds/dx_4 and dl/ds have to be replaced by the expressions given by (4) and (5) which were derived from the equations of the geodesic lines. Further we write dt instead of dx_4 and $r\,d\varphi$ instead of dl. The final result is

$$\text{(7a)}\quad \begin{cases} T_{11} = T_{22} = \tfrac{1}{2}mna^{-\frac{1}{2}}\dfrac{\beta'}{\alpha'}\left(\dfrac{\alpha'}{\alpha'-\beta'}\right)^{\frac{1}{2}}, \\[2ex] \dfrac{a}{b}\,T_{44} = mna^{-\frac{3}{2}}\left(\dfrac{\alpha'}{\alpha'-\beta'}\right)^{\frac{1}{2}}, \end{cases}$$

where α and β denote the expressions

$$\text{(7b)} \qquad \begin{cases} \alpha = \ln (r^2 a), \\ \beta = \ln b. \end{cases}$$

3. The Differential Equations of the Gravitational Field

The differential equation of a gravitational field which is due to a matter-energy tensor are

$$\text{(8)} \qquad G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \kappa T_{\mu\nu} = 0.$$

These equations have to be specialized for a static field of the type (2). By a straight forward calculation the following equations are obtained for a point on the x_3-axis:

$$\text{(9)} \qquad -G_{33} = \frac{a'}{ra} + \frac{b'}{rb} + \frac{1}{4}\left(\frac{a'}{a}\right)^2 + \frac{1}{2}\frac{a'}{a}\frac{b'}{b} = 0,$$

$$\text{(10)} \qquad G_{11} = -\frac{1}{2}\left(\frac{a'}{a}\right)' - \frac{1}{2}\left(\frac{b'}{b}\right)' - \frac{1}{2}\frac{a'}{ra} - \frac{1}{2}\frac{b'}{rb} - \frac{1}{4}\left(\frac{b'}{b}\right)^2 + \kappa T_{11} = 0,$$

$$\text{(11)} \qquad \frac{a}{b} G_{44} = \left(\frac{a'}{a}\right)' + 2\frac{a'}{ra} + \frac{1}{4}\left(\frac{a'}{a}\right)^2 + \kappa T_{44}\frac{a}{b} = 0.$$

For T_{11} and T_{44} we have to substitute the expressions given by (7a), (7b). As m is to be considered a given constant, the only functions of the coördinates in these equations are n, a, and b. It is to be expected in the first place that n, i.e. radial distribution of matter, remains undetermined by the equations. This makes necessary the existence of an identity between the equations (9), (10), (11). In fact such an identity exists. Its form is

$$\text{(12)} \qquad 0 \equiv G'_{33} + \left(\frac{2}{r} + \frac{1}{2}\frac{b'}{b}\right) G_{33} - \left(\frac{2}{r} + \frac{a'}{a}\right) G_{11} + \frac{1}{2}\frac{b'}{b} G_{44}.$$

It may be obtained in the following way: We have constructed $T_{\mu\nu}$ by considering particles which satisfy the equations of motion in the field. Therefore the covariant divergence of this tensor is bound to vanish identically. On the other hand, the divergence of $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ vanishes identically on account of the Bianchi identities. Of these four equations having the form of divergences only the one with the index 3 yields anything which does not already vanish identically with respect to the $G_{\mu\nu}$, and that is (12). From the form of (12) it follows that (10) is the consequence of (9) and (11). The problem is therefore reduced to (9) and (11), and the particle density remains undetermined, as was to be expected.

This result makes possible a further simplification of the problem. If, in (9), the quantities $\alpha = \ln (r^2 a)$ and $\beta = \ln b$ are introduced, we obtain the equation

$$\text{(13)} \qquad -\frac{2}{r^2} + \frac{1}{2}\alpha'^2 + \alpha'\beta' = 0.$$

By taking into account (13) and (7a), we obtain from (11)

$$\alpha'' + \frac{\alpha'}{r} + \frac{1}{4}\alpha'^2 - \frac{1}{r^2} + \kappa mna^{-\frac{1}{2}}\left(\frac{\alpha'^2}{\frac{3}{2}\alpha'^2 - \frac{2}{r^2}}\right)^{\frac{1}{2}} = 0. \tag{14}$$

This is a differential equation for a alone. When a is already known b is obtained by a simple integration from

$$\beta' = \frac{1}{\alpha'}\left(\frac{2}{r^2} - \frac{1}{2}\alpha'^2\right). \tag{13a}$$

4. Localization of the Particles within a Thin Spherical Shell

Outside the cluster, the gravitational field is represented by Schwarzschild's solution which, with our choice of the coördinate system, is given by (1). Inside the cluster, the field is determined by (14). Thereby, the function n is to be considered as given. However, n is not completely arbitrary, as the total radius of the cluster is restricted by the lower limit given by (6a).

Equation (14) represents a complicated relation between the particle density n and the function a representing the gravitational field. The limiting case, however, in which the gravitating particles are concentrated within an infinitely thin spherical shell, between $r = r_0 - \Delta$ and $r = r_0$, is comparatively simple. Of course, this case could only be realized if the individual particles had the rest-volume zero, which cannot be the case. This idealization, however, still is of interest as a limiting case for the radial distribution of the particles.

We divide the whole space into three zones for separate consideration, part O to be the part outside the shell, $r \geqq r_0$, part I to be the part inside the shell, $r \leqq r_0 - \Delta$, and part S to be the part of the shell $r_0 - \Delta \leqq r \leqq r_0$. In O, the gravitational field is represented by (1), in I, it is represented by (2) with constant values of a and b. It follows that a' (and α') have to change within S the faster the smaller Δ is chosen. However, as a' remains finite in S, a itself changes only infinitely little in S. It is, therefore, permissible in S to neglect α' compared with α''. We therefore replace (14) within S by

$$\alpha'' + \kappa mna^{-\frac{1}{2}}\left(\frac{\alpha'^2}{\frac{3}{2}\alpha'^2 - \frac{2}{r^2}}\right)^{\frac{1}{2}} = 0, \tag{14a}$$

where a and r are to be treated as constants for integration purposes. We introduce the variable

$$z^2 = \tfrac{3}{4}\, r^2\alpha'^2 - 1$$

and the "constant"

$$C = \kappa ma^{-\frac{1}{2}}\, \frac{r}{\sqrt{2}}$$

and obtain the equation

(14b)
$$\left(1 - \frac{1}{1 + z^2}\right) dz = Cn\,dr.$$

z is hereby determined as a function of r within S if n is given as a function of r. When the integration is carried out between $r_0 - \Delta$ and r_0 we obtain

(15)
$$\left| z - \operatorname{arctg} z \right|_{r_0}^{r_0-\Delta} = \frac{C}{4\pi r_0^2} N = \frac{\kappa}{8\pi} \sqrt{2}\, a^{-\frac{1}{2}} \frac{mN}{r_0},$$

where N designates the number of particles in S. It follows from (1) that for $r = r_0$

(15a)
$$z_{r_0} = \sqrt{2}\, \frac{(1 - 4\sigma + \sigma^2)^{\frac{1}{2}}}{1 + \sigma}, \qquad \sigma = \frac{\mu}{2r_0},$$

and from (2) that, because of a and b being constant in I, in I

(15b)
$$z_{r_0 - \Delta} = \sqrt{2}.$$

It follows from (6a) that

$$\sigma < \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}.$$

It turns out that this is just the condition for the numerator of the expression for z_{r_0} to be real. (15), for each possible r_0, gives the relationship between the sum of the masses of the particles, mN, and the total gravitating mass μ of the cluster. For large values of r_0, with a fixed value of μ, one obtains in the limit

(16)
$$\mu = \frac{\kappa}{8\pi} mN.$$

The factor $\kappa/8\pi$ is due to the fact that m is measured in grams, μ, however, in gravitational units. (16) therefore simply states that in this limiting case the gravitating mass of the cluster is equal to the sum of the particle masses.

The most illuminating way to express this result is the following:

Outside the shell ($r \geqq r_0$), the gravitational field is given by

$$ds^2 = -\left(1 + \frac{\mu}{2r}\right)^4 (dx_1^2 + dx_2^2 + dx_3^2) + \frac{1 - \frac{\mu}{2r}}{1 + \frac{\mu}{2r}}\, dt^2.$$

Inside the shell it is given by the same expression, with the difference, however, that r is to be replaced by the constant r_0, whereby the inequality

$$r_0 > \frac{\mu}{2} (2 + \sqrt{3})$$

must be satisfied. The number N of particles of the mass m which together form the shell is given by the following consideration: As an abbreviation we introduce

$$\Sigma = \frac{\kappa}{8\pi}\frac{mN}{2r_0} = \frac{M}{2r_0}, \qquad \sigma = \frac{\mu}{2r_0}.$$

Then we have

$$\Sigma = \phi(\sigma) = \frac{[(\sqrt{2} - \operatorname{arctg}\sqrt{2}) - (z_{r_0} - \operatorname{arctg} z_{r_0})](1+\sigma)^2}{\sqrt{8}},$$

where

$$z_{r_0} = \sqrt{2}\,\frac{(1 - 4\sigma + \sigma^2)^{\frac{1}{2}}}{1+\sigma}.$$

σ can assume values between 0 and $2 - \sqrt{3}(\sim\cdot 27)$. The quantity

$$\frac{\Sigma - \sigma}{\sigma}$$

is only very little different from zero in this whole region. A few typical values are given in the following table:

σ	$\frac{\Sigma - \sigma}{\sigma}$
.05	.042
.14	.06
.2	.055
.23	.013
.27	−0.022

This leads to a very interesting consequence: First it is clear that $(\Sigma - \sigma)/\sigma$ may be replaced by $(\Sigma - \sigma)/\Sigma$ with good approximation and this by $(M - \mu)/M$. This latter quantity is the relative decrease of energy of the cluster when it contracts from an infinite radius to the radius r_0. The table shows that this contraction energy has a maximum near $\sigma = 0.15$, and for greater values of σ, i.e. smaller values of r_0, it decreases again. The physical cause of this effect is that, with decreasing r_0, the potential energy of the cluster decreases, but the kinetic energy increases. For sufficiently small values of r_0 the latter effect surpasses the former.

It is therefore clear that the decrease of the radius with decreasing energy would come to an end for a value of about $\sigma = 0.15$, i.e. a radius of about $6.7(\mu/2r_0)$, while the lower limit of the radius as given by the velocity of light is $(2 + \sqrt{3})\ (\mu/2r_0)$. The value of r corresponding to the minimum energy means an upper limit for the particle velocity in the direction of the tangent of about 0.65 times the light velocity.

5. Qualitative Discussion of the Case of Arbitrary Radial Mass Distribution

We consider the case of a given mass μ and a shell radius r_0 satisfying the inequality (6a). When a number N of particles is brought into this shell zone, as determined by (15), then the exterior gravitational field is just completely screened off from the interior I so that there the field will be Euclidean. This means that the line element in I is characterized by constant values of a and b, where b cannot reach its lower limit $1/\sqrt{3}$.

If, however, the number of particles in S is chosen smaller than according to (15) then the field will not be screened off entirely (μ is hereby regarded as being kept fixed). We can then satisfy the theory formally by replacing the Euclidean line element in I by a Schwarzschild line element of the form

$$a = A\left(1 + \frac{\mu_1}{2r}\right)^4, \qquad b = B\left(\frac{1 - \frac{\mu_1}{2r}}{1 + \frac{\mu_1}{2r}}\right)^2,$$

where A, B, and μ_1 are constants. μ_1 will be smaller than μ which characterizes the field outside the shell. This interior field has a singularity of the Schwarzschild type ($b = 0$) at $r = \mu_1/2$.

This singularity, however, can be removed by introducing a second shell S_1 inside S, which has to be constructed so that the gravitational field in its interior will be Euclidean. The whole cluster will then consist of two shells S and S_1 and will have no Schwarzschild singularity.

Again this system can be modified by reducing the number of particles in S_1 so that it will not screen off *its* exterior field (between S and S_1) entirely; then a third shell S_2, of still smaller radius, may be constructed so that *its* exterior field is just screened off entirely from its interior.

This method can be reiterated up to the center of the cluster. Thus one obtains clusters with the most varied radial mass distributions. There will be also various steady distributions. It is impossible, however, that b should vanish anywhere. The radius of the cluster will always be greater than the limiting radius $\frac{1}{2}\mu(2 + \sqrt{3})$, and it will not be possible to concentrate the matter of the cluster arbitrarily densely near the center of the cluster.

6. The Case of Continuous Particle Density

The consideration given in part 5. leads toward the solution for continuous distributions of the particle density. We divide the interval $0 \leqq r \leqq r_0$ into an infinite number of equal parts dr. We imagine that there is constructed in the center of each partition dr a shell of a two dimensional character of the type discussed in part 4. The shells may be chosen so that they are equivalent to a continuous distribution of mass. Between any two subsequent shells we shall have a gravitational field of the Schwarzschild type

$$ds^2 = -A\left(1 + \frac{\tau}{2r}\right)^4 (dx_1^2 + dx_2^2 + dx_3^2) + B\left(\frac{1 - \frac{\tau}{2r}}{1 + \frac{\tau}{2r}}\right)^2 dt^2, \tag{17}$$

where A, B, and τ are constants which differ only infinitesimally for two neighboring regions. Then the sum total of all these partial solutions constitutes the gravitational field inside the cluster. Our task is to determine A, B, and τ as functions of r.

We consider two neighboring Schwarzschild solutions which belong to the radius intervals $r - \frac{1}{2}dr$ to $r + \frac{1}{2}dr$ and $r + \frac{1}{2}dr$ to $r + \frac{3}{2}dr$. In the first region the values of A, B, and τ belong to the value r of the radius, in the second to the value $r + dr$. If we use the quantities introduced by (2) then the two local solutions are given by

$$a(r; A, \tau), \qquad a(r; A + dA, \tau + d\tau),$$

and

$$b(r; B, \tau), \qquad b(r; B + dB, \tau + d\tau),$$

where a, b are functions of r in accordance with (17). These two solutions are to assume the same values for a and b in the point $r + \frac{1}{2}dr$ because these quantities must not change when we pass through a shell occupied by particles. It follows, up to quantities of the first order

$$\frac{\partial a}{\partial A}dA + \frac{\partial a}{\partial \tau}d\tau = 0,$$

$$\frac{\partial b}{\partial B}dB + \frac{\partial b}{\partial \tau}d\tau = 0,$$

or, in accordance with (17)

$$\text{(18)} \qquad \begin{cases} \dfrac{dA}{A} + \dfrac{4}{r}\dfrac{r\,d\sigma + \sigma\,dr}{1 + \sigma} = 0, \\[2ex] \dfrac{dB}{B} - \dfrac{4}{r}\dfrac{r\,d\sigma + \sigma\,dr}{(1 + \sigma)(1 - \sigma)} = 0, \end{cases}$$

where σ is written for $\tau/2r$.

These equations determine A, B as functions of r when τ or σ is given as function of r. It turns out that α, β, computed from the solutions A, B of (18), are the solutions of (13), represented with the help of the "parameter" function σ. τ is arbitrary within certain limits because it is closely connected with the mass distribution. On the other hand, A, B, and τ have to satisfy the condition that (17) makes possible circular particle paths for all values of r, i.e. a and b have to satisfy the inequality (6). In connection with (17) we obtain the inequality

$$\text{(19)} \qquad 1 - \frac{\dfrac{B'}{B} - 4\dfrac{\sigma'}{(1 + \sigma)(1 - \sigma)}}{\dfrac{A'}{A} + \dfrac{2}{r} + 4\dfrac{\sigma'}{1 + \sigma}} > 0.$$

(18) and (19) together completely determine the problem within the cluster; σ is arbitrary save for the only restriction that, together with the values of A and B, calculated from (18), it has to satisfy (19).

For $r \geqq r_0$ we have, of course, $A = B = 1$, with $\tau = \text{const.} = \mu$.

By using (18) we may write (19) thus:

$$1 - \frac{4\,\dfrac{\sigma}{(1+\sigma)(1-\sigma)}}{2 - 4\,\dfrac{\sigma}{1+\sigma}} > 0$$

or, with some transformations:

$$\text{(19a)} \qquad \frac{(\sigma - 2 + \sqrt{3})(\sigma - 2 - \sqrt{3})}{(1-\sigma)^2} > 0.$$

This inequality has to hold within as well as outside the cluster. For infinite values of r, σ vanishes. Further σ has to be positive, as negative masses are excluded. Because of the denominator, σ can nowhere be greater than 1. Therefore the numerator of the left hand side has to be positive. As the second factor of the numerator is always negative the first factor has to be negative, too. We therefore obtain

$$\text{(19b)} \qquad \sigma < 2 - \sqrt{3}.$$

This is a generalization of (6a) as (6a) was only proven to hold for the outside boundary of the cluster.

τ represents the mass enclosed by the spherical surface of the radius r. In order that negative masses should be ruled out it is necessary that everywhere

$$\text{(20)} \qquad \frac{d\tau}{dr} \geqq 0.$$

It is further necessary that τ vanishes for $r = 0$. Save for this condition τ may be chosen arbitrarily if only σ satisfies (19b). When τ and therefore σ is given then the problem of determining the gravitational field of the form (17) is reduced to the carrying out of two integrations, according to (18).

The equations (18) give us the integration of (13) with arbitrary mass density distribution, where the latter is expressed by τ or σ. (14) gives the correspondign particle density n. We shall express n in terms of σ. We have

$$\text{(21)} \qquad 0 = \frac{2}{r}\left(\frac{1-\sigma}{1+\sigma}\right)' - \frac{4}{r^2}\frac{\sigma}{(1+\sigma)^2} + \kappa m n a^{-\frac{1}{2}} \frac{1-\sigma}{\sqrt{1 - 4\sigma + \sigma^2}}$$

together with the relations

$$\text{(22)} \qquad a = A(1+\sigma)^4, \qquad \frac{A'}{A} = -\frac{4}{r}\frac{r\sigma' + \sigma}{1+\sigma}.$$

Therefore, when σ is given as a function of r we obtain n by carrying out one integration only.

σ is positive and stays below the limit $2 - \sqrt{3}$. The square root of the denominator of the third term in (21) therefore is always positive. We further have $\tau/2r$ where τ is the gravitating mass contained in a sphere of the radius r. τ therefore increases monotonically with increasing r. If the mass density is to be finite in the region around $r = 0$ then τ has to decrease in that region at least as fast as r^3 and σ at least as fast as r^2. Under these conditions the two first terms in (21) will be finite everywhere, and also A'/A, A, and a. (21) therefore gives us a finite value for n. It is further possible to prove from the properties of τ that the sum of the two first terms in (21) is negative everywhere.

From all these considerations it can be followed that a and b are finite and not zero in the whole space.

By combining (2), (4), (17), and (18) one can show that the ratio V between the particle velocity and between the light velocity pointing into the same direction, is given by

$$V^2 = \frac{\beta'}{\alpha'} = \frac{2\sigma}{(1-\sigma)^2}. \tag{23}$$

When σ stays below a given limit V will stay below a certain limit, too.

7. A Special Case of Continuous Mass Distribution

It is of some interest to investigate the case where σ inside the cluster is a constant σ_0. Strictly speaking this case falls outside of our conditions as σ ought to decrease toward the point $r = 0$ at least as fast as r^2 in order that the density in the neighborhood of the center should stay finite. We can satisfy this condition by choosing σ for instance

$$\sigma = \sigma_0(1 - e^{-cr^2}) \tag{24}$$

where c is to be an arbitrary constant. We then consider from the start the limiting case of $c = \infty$. This special case is discussed here in order to supplement the discussions of part 4. There the whole mass was distributed as far outside (within the total radius r_0) as possible, while here we have a strong concentration of mass toward the center of the cluster.

As τ is the gravitating mass enclosed by a spherical surface of the radius r, $d\tau/(4\pi r^2 dr)$ is the mean density of the gravitating mass in the point r. As $\tau = 2\sigma_0 r$ we obtain for this mean density $\sigma_0/2\pi r^2$, i.e. a radial decrease of the density like $1/r^2$ up to the cluster boundary $r = r_0$.

From (18), in accordance with (24) (in the limiting case of vanishing expo nential term), we obtain

$$\begin{cases} \dfrac{dA}{A} = -\dfrac{4\sigma_0}{1+\sigma_0}\dfrac{dr}{r}, \\[2ex] \dfrac{dB}{B} = \dfrac{4\sigma_0}{1-\sigma_0^2}\dfrac{dr}{r}, \end{cases} \tag{18a}$$

and since for $r = r_0$, A and B have to assume the value 1

$$\text{(18b)} \qquad \begin{cases} A = \left(\dfrac{r}{r_0}\right)^{-4\sigma_0/(1+\sigma_0)}, \\ B = \left(\dfrac{r}{r_0}\right)^{4\sigma_0/(1-\sigma_0^2)}. \end{cases}$$

For $r = 0$ we obtain $a = \infty$ and $b = 0$. This type of singularity, however, is not to be taken seriously because it would be avoided if we had taken into consideration the exponential term in (24). It is to be noted that through a suitable choice of the mass distribution this singularity can be approximated, but not reached.

We make use of (21) in order to determine the relation existing between the sum of the rest masses of the particles M

$$M = \frac{\kappa}{8\pi} m \int_0^{r_0} n \cdot 4\pi r^2 \, dr,$$

and the total gravitating mass of the cluster μ. It can be shown that the first term of (21) gives only a vanishing contribution for infinitely great values of c. This follows from the fact that $\left(\frac{1 - \sigma}{1 + \sigma}\right)'$ vanishes everywhere where the influence of the exponential term of (24) has become unnoticeable. We compute the contribution of the second term in (21) by omitting the exponential term from the start and obtain, after a short calculation, as the final result, with $\mu = 2r_0\sigma_0$

$$\text{(25)} \qquad M = \mu(1 - 4\sigma_0 + \sigma_0^2)^{\frac{1}{2}} \frac{1 + \sigma_0}{(1 - \sigma_0)^2}.$$

This equation when compared with the relation

$$\text{(26)} \qquad \mu = 2\sigma_0 r_0$$

allows an easy discussion of the essential properties of clusters of this type.

First it is easy to see that we have extremely simple relations when we change M but keep fixed σ_0 $(0 < \sigma_0 < 2 - \sqrt{3})$ and thereby the tangential velocity of the particles as measured in light velocity units. When M is multiplied by z the gravitating mass will be $z\mu$ and the diameter of the cluster will be $z \cdot 2r$. The mean density will be multiplied by $1/z^2$.

In order to obtain a survey of all possibilities it is therefore sufficient to keep fixed the number of constituting particles and thereby M and to vary σ_0 together with the diameter $2r_0$ and the gravitating mass μ. We obtain for $M = 1$

$$\mu = \frac{(1 - \sigma_0)^2}{1 + \sigma_0} (1 - 4\sigma_0 + \sigma_0^2)^{-\frac{1}{2}}.$$

The following table gives μ and $2r_0$ for $M = 1$ as functions of σ_0 (approximately):

σ_0	μ	$2r_0$
0.	1.	∞
.05	.988	19.76
.1	.948	9.48
.15	.97	6.56
.2	1.13	5.65
.23	1.32	5.63
.25	1.82	7.40
.26	2.63	10.1
.268	∞	∞

When the cluster is contracted from an infinite diameter its mass decreases at the most about 5%. This minimal mass will be reached when the diameter $2r_0$ is about 9. The diameter can be further reduced down to about 5.6, but only by adding enormous amounts of energy. It is not possible to compress the cluster any more while preserving the chosen mass distribution. A further addition of energy enlarges the diameter again. In this way the energy content, i.e. the gravitating mass of the cluster, can be increased arbitrarily without destroying the cluster. To each possible diameter there belong two clusters (when the number of particles is given) which differ with respect to the particle velocity.

Of course, these paradoxical results are not represented by anything in physical nature. Only that branch belonging to smaller σ_0 values contains the cases bearing some resemblance to real stars, and this branch only for diameter values between ∞ and $9M$.

The case of the cluster of the shell type, discussed earlier in this paper, behaves quite similarly to this one, despite the different mass distribution. The shell type cluster, however, does not contain a case with infinite μ, given a finite M.

The essential result of this investigation is a clear understanding as to why the "Schwarzschild singularities" do not exist in physical reality. Although the theory given here treats only clusters whose particles move along circular paths it does not seem to be subject to reasonable doubt that more general cases will have analogous results. The "Schwarzschild singularity" does not appear for the reason that matter cannot be concentrated arbitrarily. And this is due to the fact that otherwise the constituting particles would reach the velocity of light.

This investigation arose out of discussions the author conducted with Professor H. P. Robertson and with Drs. V. Bargmann and P. Bergmann on the mathematical and physical significance of the Schwarzschild singularity. The problem quite naturally leads to the question, answered by this paper in the negative, as to whether physical models are capable of exhibiting such a singularity.

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